Lecture 4: The Linear Time Selection

Selection Problem

Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, and an integer i, $1 \leq i \leq n$, find the ith smallest element. When $i = \lceil n/2 \rceil$, it is called the median problem.

Example: Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

Question: How do you solve this problem?

First Solution: Selection by sorting

Step 1: Sort the elements in ascending order with any algorithm of complexity $O(n \log n)$.

Step 2: Return the ith element of the sorted array.

The complexity of this solution is $O(n \log n)$.

Question: Can we do better?

Answer: YES, but we need to recall Partition(A, p, r) used in Quicksort!

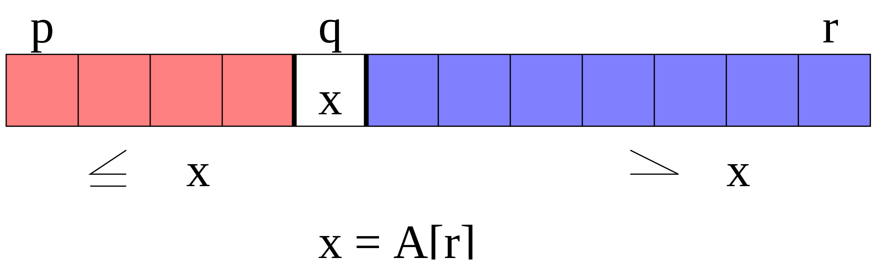
Second Solution : Linear running time in average

Recall of Partition(A, p, r)

Definition: Rearrange the array $A[p..r]$ into two (possibly empty) subarrays $A[p..q-1]$ and $A[q+1..r]$ such that

$$A[u] \leq A[q] < A[v]$$

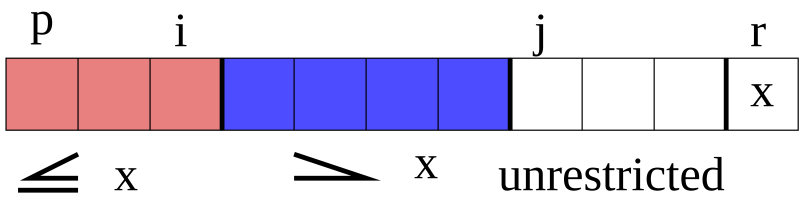
for any $p \leq u \leq q-1$ and $q+1 \leq v \leq r$.



(1) The original $A[r]$ is used as the pivot.
(2) It is a deterministic algorithm.
(3) The element for the qth position is found!

Note that this partition is different from the partition we used in COMP 171.

$\boxed{\text{The Idea of Partition}(A, p, r)}$

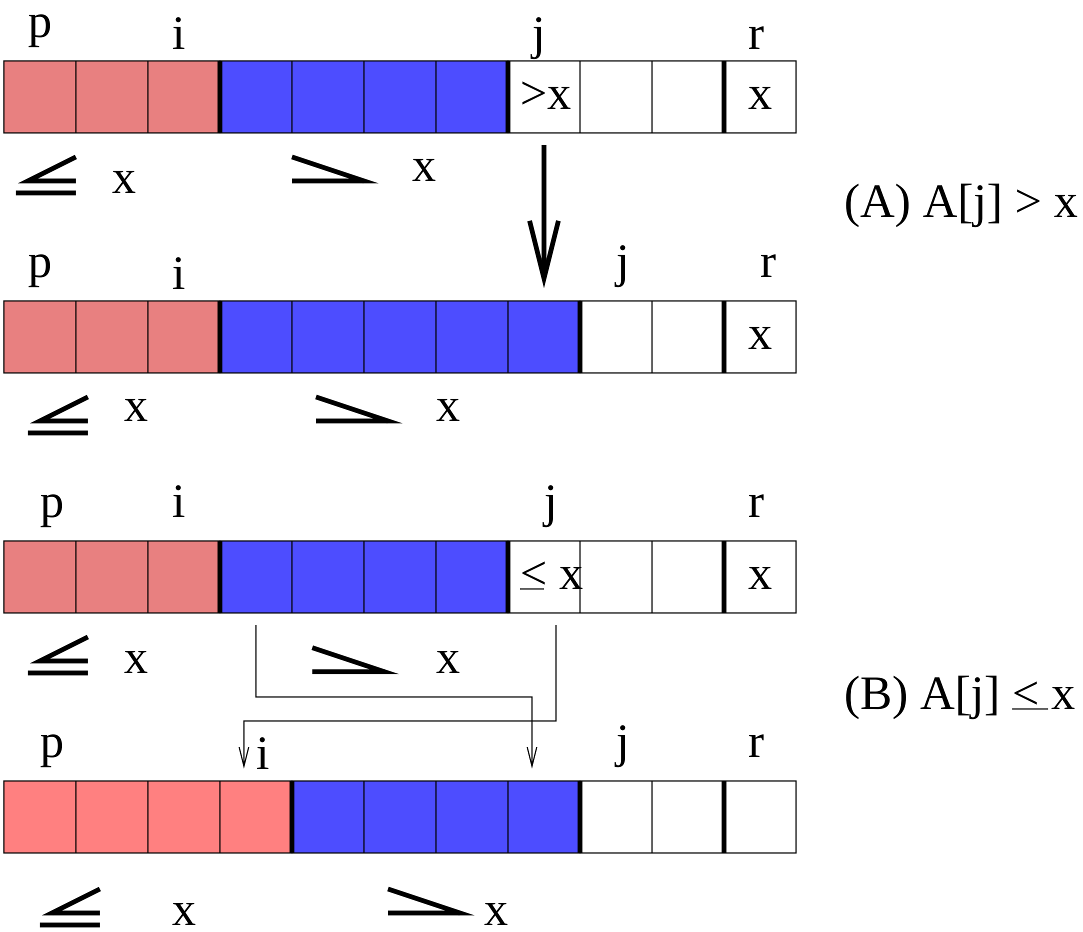


(1) Initially $(i, j) = (p - 1, p)$.

(2) Increase j by 1 each time to find a place for $A[j]$.
At the same time increase i when necessary.

(3) The procedure stops when $j = r$.

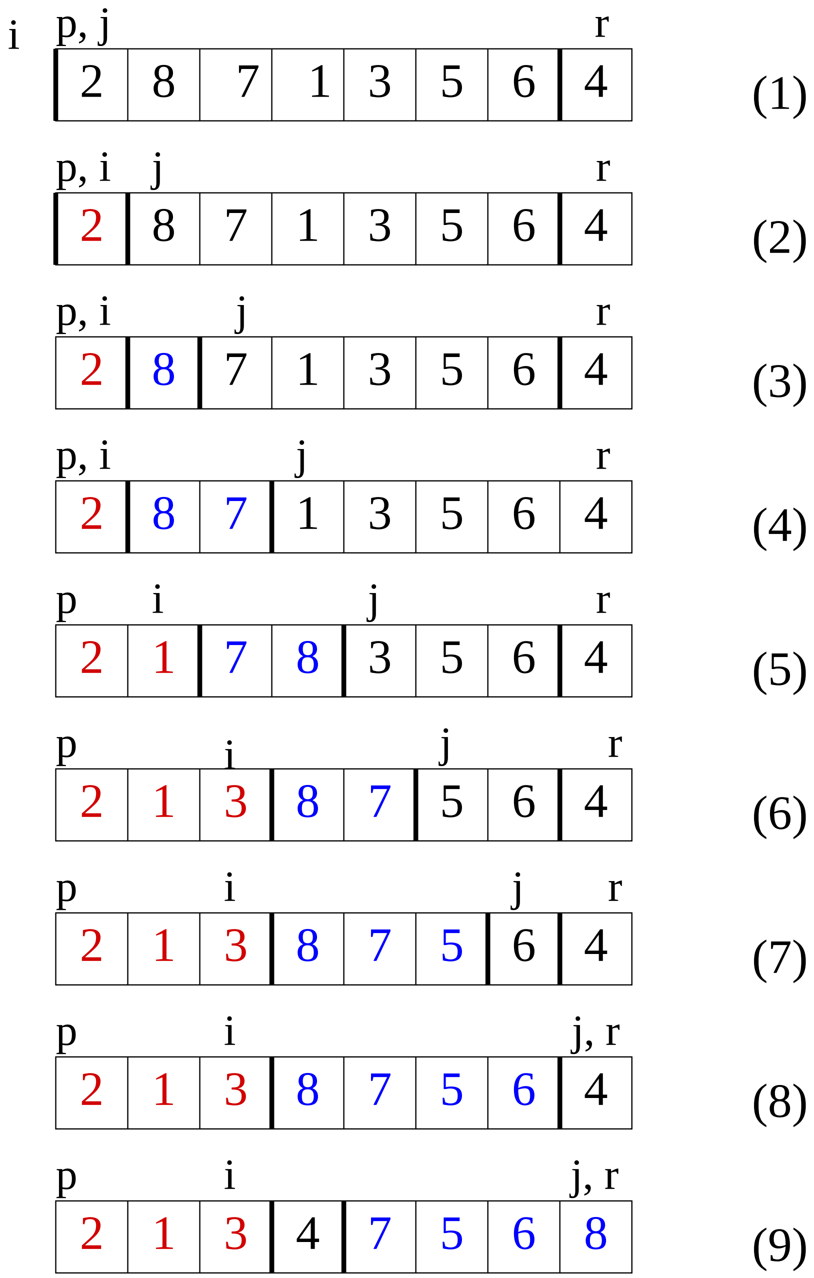
One Iteration of the Procedure Partition



(A) Only increase j by 1.

(B) $i \leftarrow i + 1$. $A[i] \leftrightarrow A[j]$. $j \leftarrow j + 1$.

The Operation of Partition(A, p, r): Example



The Partition(A, p, r) Algorithm

```
Partition(A, p, r)
{
 // A[r] is the pivot element
 x = A[r];
 i = p-1;
 for (j = p to r-1) {
  if (A[j] <= x) {
   i = i+1;
   exchange A[i] and A[j]
  }
 }

 // put pivot in position
 exchange A[i+1] and A[r]
 // q = i+1
 return i+1;
}
```

The Running Time of Partition(A, p, r)

comparison of array elements

assignment, addition, comparison of loop variables

Partition(A, p, r):

$x = A[r]$	1
$i = p - 1$	1
for $j = p$ to $r - 1$	$2(r - p)$
if $A[j] \le x$	$(r - p)$
$i = i + 1$	$\le (r - p)$
exchange $A[i] \leftrightarrow A[j]$	$\le 3(r - p)$
exchange $A[i + 1] \leftrightarrow A[r]$	3
return $i + 1$	1

Total: $(r - p)$ and $\le \{6(r - p) + 6\}$

Running time is $\Theta(r - p)$, that is, linear in the length of the array $A[p..r]$.

Randomized-Partition(A, p, r)

The Idea: In the algorithm Partition(A, p, r), $A[r]$ is always used as the pivot x to partition the array $A[p..r]$.

In the algorithm Randomized-Partition(A, p, r), we randomly choose an j, $p \leq j \leq r$, and use $A[j]$ as pivot.

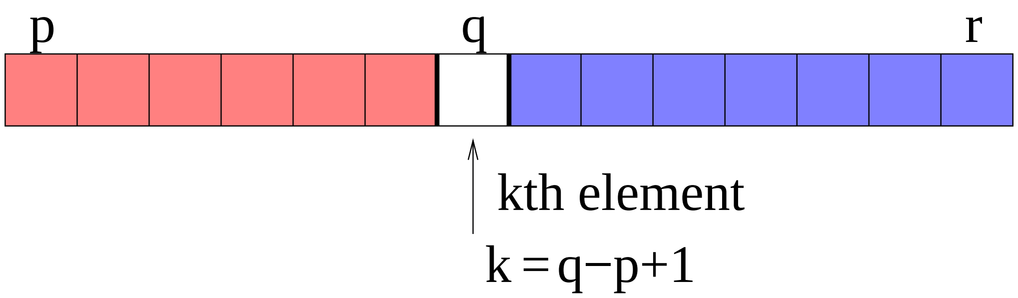
```
Randomized-Partition(A, p, r)
{
 j = random(p, r);
 exchange A[r] and A[j]
 Partition(A, p, r);
}
```

Remark: `random(p, r)` is a pseudorandom-number generator that returns a random number between p and r.

$\boxed{\textbf{Randomized-Select}(A, p, r, i),\ 1 \le i \le r - p + 1}$

Problem: Select the ith smallest element in $A[p..r]$, where $1 \le i \le r - p + 1$.

Solution: Apply Randomized-Partition(A, p, r), getting



Case 1: $i = k$, pivot is the solution.

Case 2: $i < k$, the ith smallest element in $A[p..r]$ must be the ith smallest element in $A[p..q-1]$.

Case 3: $i > k$, the ith smallest element in $A[p..r]$ must be the $(i-k)$th smallest element in $A[q+1..r]$.

If necessary, recursively call the same procedure to the subarray.

Randomized-Select(A, p, r, i), $1 \le i \le r - p + 1$

if $p == r$
    return $A[p]$
$q =$ Randomized-Partition(A, p, r)
$k = q - p + 1$
if $i == k$ the pivot is the answer
    return $A[q]$
else if $i < k$
    return Randomized-Select$(A, p, q - 1, i)$
else
    return Randomized-Select$(A, q + 1, r, i - k)$

Remark: To find the ith smallest element in $A[1..n]$, call Randomized-Select($A, 1, n, i$).

Running Time of Randomized-Select($A, 1, n, i$)

Let $T(n, i)$ be the average number of comparisons of array elements for $1 \leq i \leq n$.

Then $T(1, 1) = 0$ and for $n > 1$ we get

$$
\begin{aligned}
T(n,i) = (n-1) & \qquad \text{initial partition} \\
+\tfrac{1}{n}\{\textstyle\sum_{k=1}^{i-1} T(n-k, i-k) & \qquad \text{recursion, } k < i \\
+\textstyle\sum_{k=i+1}^{n} T(k-1, i)\} & \qquad \text{recursion, } k > i
\end{aligned}
$$

We will prove by induction on n that

$$T(n, i) < 4\,n$$

for all n and i.

Proof that $T(n,i) < 4\,n$

Induction basis: $T(1,1) = 0 < 4 \cdot 1$.
Induction step: Assume that $T(m,j) < 4\,m$ for all $m < n$ and $1 \le j \le m$. Then

$$
\begin{aligned}
& T(n,i) \\
&= n - 1 + \frac{1}{n}\{\sum_{k=1}^{i-1} T(n-k, i-k) + \sum_{k=i+1}^{n} T(k-1, i)\} \\
&< n - 1 + \frac{1}{n}\{\sum_{k=1}^{i-1} 4(n-k) + \sum_{k=i+1}^{n} 4(k-1)\} \\
&= n - 1 + \frac{1}{n}\{4n(i-1) - 4\frac{i(i-1)}{2} + 4\frac{n(n-1)}{2} - 4\frac{i(i-1)}{2}\} \\
&= n - 1 + \frac{1}{n}\{2n^2 - 6n + (4n+4)i - 4i^2\}.
\end{aligned}
$$

$\boxed{\textbf{Proof that } T(n,i) < 4\,n}$

$$T(n,i) < n - 1 + \frac{1}{n} f(i),$$

where

$$f(x) = 2n^2 - 6n + (4n + 4)x - 4x^2.$$

$$f'(x) = (4n + 4) - 8x = 0$$

$$f''(x) = -8 < 0$$

for $x = (n + 1)/2$. Hence

$$f(x) \leq f((n + 1)/2) = 3n^2 - 4n + 1$$

for all x. Therefore

$$T(n,i) \leq n - 1 + 3n - 4 + \frac{1}{n} < 4n.$$

$\boxed{\textbf{Running Time of Randomized-Select}(A, 1, n, i)}$

We proved that $T(n, i) < 4n$. Since $T(n, i) \geq n - 1$, we have in particular that

$$T(n, i) = \Theta(n).$$

Randomized-Quicksort Algorithm

We make use of the Randomized-Partition idea to develop a new version of quicksort.

```
Randomized-Quicksort(A, p, r)
{
 if (p < r) {
  q = Randomized-Partition(A, p, r);
  Randomized-Quicksort(A, p, q-1);
  Randomized-Quicksort(A, q+1, r);
 }
}
```

Does it run faster than the original version of quicksort?

Running Time of the Randomized-Quicksort

Results :

Worst Case: $T(n) = \Theta(n^2)$.
Average Case: $T(n) = O(n \log n)$.

Clearly, the worst case is still $\Theta(n^2)$, what about the average case?

Average running time of Randomized-Quicksort

Key observations:

- The running time of (randomized) quicksort is dominated by the time spent in (randomized) partition. In the partition procedure, the time is dominated by the *number of key comparisons*.

- When a pivot is selected, the pivot is compared with every other elements, then the elements are partitioned into two parts accordingly.

- Elements in different partition are NEVER compared with each other in *all* operations.

Tricks: We find the *expected* number of comparisons for **all** randomized-partition calls.

Average running time of Randomized-Quicksort

Let A be the input array which is a permutation of the n distinct elements $z_1 < z_2 < \ldots < z_n$.

Let X be the total number of comparisons performed in ALL calls to randomized-partition. Let X_{ij} be the number of comparisons between z_i and z_j, observe that X_{ij} can only be 0 or 1. Our goal is to compute the expected value of X, i.e.,

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [Pr\{z_i \text{ is compared to } z_j\} \times 1 \\
&\quad + Pr\{z_i \text{ is not compared to } z_j\} \times 0] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr\{z_i \text{ is compared to } z_j\}
\end{aligned}$$

Average running time of Randomized-Quicksort

It remains to show how to find $Pr\{z_i$ is compared to $z_j\}$.

For $1 \leq i \leq j \leq n$, let $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$ (remember $z_i < z_{i+1} < \ldots < z_j$).

Key observations:

- If z_i or z_j is selected as a pivot BEFORE any elements in $\{z_{i+1}, z_{i+2}, \ldots, z_{j-1}\}$, z_i and z_j will be compared.

- Conversely, if any element in Z_{ij} other then z_i or z_j is selected as a pivot before z_i and z_j, z_i and z_j will be placed in DIFFERENT partitions, and hence they will NOT compare with each other in ALL randomized-partition calls.

- ANY element other than the elements in Z_{ij} has no effect to $Pr\{z_i$ is compared to $z_j\}$.

Average running time of Randomized-Quicksort

It remains to find the probability that z_i or z_j is the first pivot chosen from Z_{ij}.

$$
\begin{aligned}
&Pr\{z_i \text{ is compared to } z_j\} \\
&\quad = Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\} \\
&\qquad\quad + Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\} \\
&\quad = \frac{1}{j-i+1} + \frac{1}{j-i+1} \\
&\quad = \frac{2}{j-i+1}
\end{aligned}
$$

Average running time of Randomized-Quicksort

Putting everything together, we have

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) \\
&= O(n \lg n)
\end{aligned}
$$

Hence, the expected number of comparisons is $O(n \lg n)$, which is the average running time of Randomized-Quicksort.