The Insertionsort Extra Credit

After performing the average case analysis of Insertionsort we asked the following extra credit question: What is the value of

$$\sum_{p=1}^{n} E(e_p)$$

where

$$e_p = \begin{cases} 
0 & \text{if } p \text{ is smaller than all items that appear to } p \text{'s left,} \\
1 & \text{otherwise.}
\end{cases}$$

In this problem we assume that the input to Insertionsort is equally likely to be any of the \(n!\) permutations of \(1, \ldots, n\).

For \(j \leq n\) set

$$X_j = \begin{cases} 
0 & \text{if } a_j \text{ is smaller than all items that appear to } a_j \text{'s left,} \\
1 & \text{otherwise.}
\end{cases}$$

Since the input is a permutation of \(1, \ldots, n\) we have that

$$\sum_{j=1}^{n} X_j = \sum_{p=1}^{n} e_p$$

so, by the linearity of the expectation operator,

$$\sum_{j=1}^{n} E(X_j) = E \left( \sum_{j=1}^{n} X_j \right) = E \left( \sum_{p=1}^{n} e_p \right) = \sum_{p=1}^{n} E(e_p).$$

The important observation now is that

\[
\Pr(X_j = 1) = 1 - \Pr(X_j = 0) = 1 - \frac{1}{j} \quad \text{(see over) so}
\]

$$E(X_j) = 1 \cdot \Pr(X_j = 1) + 0 \cdot \Pr(X_j = 0) = 1 - \frac{1}{j}.$$  

Thus

$$\sum_{p=1}^{n} E(e_p) = \sum_{j=1}^{n} E(X_j) = \sum_{j=1}^{n} \left( 1 - \frac{1}{j} \right) = n - \sum_{j=1}^{n} \frac{1}{j}.$$  

Finding \(\sum_{p=1}^{n} E(e_p) = n - \sum_{j=1}^{n} \frac{1}{j}\) was enough to get the extra credit correct.

To go further note that the sum \(\sum_{j=1}^{n} \frac{1}{j}\) is usually denoted by \(H_j\) and is called the \(n^{th}\) Harmonic number; it is known that \(H_n = \ln n + O(1)\) (page 1060 in CLRS). Thus

$$\sum_{p=1}^{n} E(e_p) = n - \ln n - O(1).$$
On the previous page we used the fact that $\Pr(X_j = 0) = \frac{1}{j}$. Why is this true?

To see that this is true note that we can partition the $n!$ permutations of $1, \ldots, n$ into different equivalency classes as follows: Two permutations $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ will be put in the same class if and only if

$$\forall i > j, \quad a_i = b_i.$$ 

(Note that this gives $n!/(n-j)!$ different classes). This means that, as sets, $\{a_1, \ldots, a_j\} = \{b_1, \ldots, b_j\}$ so each class is precisely identified by the $j$ first items in its permutations. Therefore each class contains exactly $j!$ permutations.

Of these $j!$ permutations exactly $(j-1)!$ have the property that their $j$th element is smaller than all items to its left (why?). This means that $1/j$ of the permutations in the class have the property that $X_j = 0$. Since this is true of every class and the classes partition the set of permutations this implies that $1/j$ of all permutations have $X_j = 0$ which is just another way of saying that $\Pr(X_j = 0) = \frac{1}{j}$.