The Insertionsort Extra Credit

After performing the average case analysis of Insertionsort we asked the following exra credit question: What is the value of

$$\sum_{p=1}^{n} E(e_p)$$

where

$$e_p = \begin{cases} 0 & \text{if } p \text{ is smaller than all items that appear to } p\text{'s left,} \\ 1 & \text{otherwise.} \end{cases}$$

In this problem we assume that the input to Insertionsort is equally likely to be any of the $n!$ permutations of $1, \ldots, n$.

For $j \leq n$ set

$$X_j = \begin{cases} 0 & \text{if } a_j \text{ is smaller than all items that appear to } a_j\text{'s left,} \\ 1 & \text{otherwise.} \end{cases}$$

Since the input is a permutation of $1, \ldots, n$ we have that

$$\sum_{j=1}^{n} X_j = \sum_{p=1}^{n} e_p$$

so, by the linearity of the expectation operator,

$$\sum_{j=1}^{n} E(X_j) = E\left(\sum_{j=1}^{n} X_j\right) = E\left(\sum_{p=1}^{n} e_p\right) = \sum_{p=1}^{n} E(e_p).$$

The important observation now is that
$\Pr(X_j = 1) = 1 - \Pr(X_j = 0) = 1 - \frac{1}{j}$ (see over) so

$$E(X_j) = 1 \cdot \Pr(X_j = 1) + 0 \cdot \Pr(X_j = 0) = 1 - \frac{1}{j}.$$

Thus

$$\sum_{p=1}^{n} E(e_p) = \sum_{j=1}^{n} E(X_j) = \sum_{j=1}^{n} \left(1 - \frac{1}{j}\right) = n - \sum_{j=1}^{n} \frac{1}{j}$$

Finding $\sum_{p=1}^{n} E(e_p) = n - \sum_{j=1}^{n} \frac{1}{j}$ was enough to get the extra credit correct.
To go further note that the sum $\sum_{j=1}^{n} \frac{1}{j}$ is usually denoted by H_j and is called the n*'th Harmonic number*; it is known that $H_n = \ln n + O(1)$ (page 1060 in CLRS). Thus $\sum_{p=1}^{n} E(e_p) = n - \ln n - O(1)$.

On the previous page we used the fact that $\Pr(X_j = 0) = \frac{1}{j}$. Why is this true?

To see that this is true note that we can partition the $n!$ permutations of $1, \ldots, n$ into different equivalency classes as follows: Two permutations $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, n_n$ will be put in the same class if and only if

$$\forall i > j, \quad a_i = b_i.$$

(Note that this gives $n!/(n-j)!$ different classes). This means that, as sets, $\{a_1, \ldots, a_j\} = \{b_1, \ldots, b_j\}$ so each class is precisely identified by the j first items in its permutations. Therefore each class contains exactly $j!$ permutations.

Of these $j!$ permutations exactly $(j-1)!$ have the property that their jth element is smaller than all items to its left (why?). This means that $1/j$ of the permutations in the class have the property that $X_j = 0$. Since this is true of every class and the classes partition the set of permutations this implies that $1/j$ of *all* permutations have $X_j = 0$ which is just another way of saying that $\Pr(X_j = 0) = \frac{1}{j}$.