Design and Analysis of Algorithms

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Comp 271

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Information about the Lecturer

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Textbook and Lecture Notes

Textbook: Cormen, Leiserson, Rivest, Stein: "Introduction to Algorithms", 2.ed. MIT Press 2001.

Lecture Slides: Available on course webpage http://www.cs.ust.hk/course/comp271

References: Recommendations

1. Dave Mount: Lecture Notes
   Available on course web page
2. Jon Bentley: *Programming Pearls (2nd ed).* Addison-Wesley, 2000.
3. Michael R. Garey & David S. Johnson: *Computers and intractability : a guide to the theory of NP-completeness.* W. H. Freeman, 1979.
4. Robert Sedgewick: *Algorithms in C++ (3rd ed) Volumes 1 and 2.* Addison-Wesley, 1998.

About COMP 271

A continuation of COMP 171, with advanced topics and techniques. Main topics are:

1. Design paradigms: divide-and-conquer, greedy algorithms, dynamic programming.

2. Analysis of algorithms (goes hand in hand with design).

3. Graph Algorithms.

4. Complexity classes (P, NP, NP-complete).

Prerequisite: Discrete Math. and COMP 171

We assume that you know

- Sorting: Quicksort, Insertion Sort, Mergesort, Radix Sort (with analysis). Lower Bounds on Sorting.
- Big $O()$ notation and simple analysis of algorithms
- Heaps
- Graphs and Digraphs. Breadth & Depth first search and their running times. Topological Sort.
- Balanced Binary Search Trees (dictionaries)
- Hashing

Tentative Syllabus

- *Introduction & Review*

- *Maximum Contiguous Subarray:*
  case study in algorithm design

- *Divide-and-Conquer Algorithms:* Mergesort, Polynomial Multiplication, Randomized Selection

- *Graphs:*
  - Review: Notation, Depth/Breadth First Search
  - Cycle Finding & Topological Sort
  - Minimum Spanning Trees: Kruskal's and Prim's algorithms
  - Dijkstra's shortest path algorithm

- *Dynamic Programming:* Knapsack, Chain Matrix Multiplication, Longest Common Subsequence, All Pairs Shortest Path

- *Greedy algorithms:* Activity Selection, Huffman Coding

- *Complexity Classes:* Nondeterminism, the classes P and NP, NP-complete problems, polynomial reductions

Other Information

- Lecture and tutorial schedule, TAs.
  No Tutorials this week.
  Tutorials start on **February 14, 2003**.

- Question banks: To help you review

- Assignments: 4, each worth 5% of grade
  Midterm: worth 35% of grade
  Final exam (comprehensive): worth 45% of grade.

- Final Grade. Will be curved based on class performance. Guaranteed Grades:
  Average of $\geq$ 95 $\Rightarrow A$
  Average of $\geq$ 85 $\Rightarrow B$
  Average of $\geq$ 75 $\Rightarrow C$
  Average of $\geq$ 65 $\Rightarrow D$

Classroom Etiquette

- No pagers and cell phones – switch off in classroom.
- Latecomers should enter **QUIETLY**.
- No loud talking during lectures.
- But please ask questions and provide feedback.

Lecture 1: Introduction

Computational Problems and Algorithms

Definition: A computational problem is a specification of the desired input-output relationship.

Definition: An instance of a problem is all the inputs needed to compute a solution to the problem.

Definition: An algorithm is a well defined computational procedure that transforms inputs into outputs, achieving the desired input-output relationship.

Definition: A correct algorithm halts with the correct output for every input instance. We can then say that the algorithm solves the problem.

Example of Problems and Instances

Computational Problem: Sorting

- **Input:** Sequence of n numbers $\langle a_1, \cdots, a_n \rangle$.

- **Output:** Permutation (reordering)

$$\langle a'_1, a'_2, \cdots, a'_n \rangle$$

such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Instance of Problem: $\langle 8, 3, 6, 7, 1, 2, 9 \rangle$

Example of Algorithm: Insertion Sort

In-Place Sort: uses only a fixed amount of storage beyond that needed for the data.

Pseudocode: A is an array of numbers

for $j = 2$ to length(A)
$\{$ key = $A[j]$;
    $i = j - 1$;
    while ($i \geq 1$ and $A[i] > \text{key}$)
    $\{$ $A[i+1] = A[i]$;
        $i = i - 1$;
    $\}$
    $A[i+1] = \text{key}$;
$\}$

Pause: How does it work?

Insertion Sort: an Incremental Approach

To sort a given array of length n, at the ith step it sorts the array of the first i items by making use of the sorted array of the first $i-1$ items in the $(i-1)$th Step.

Example: Sort $A = \langle 6, 3, 2, 4 \rangle$ with Insertion Sort.

Step 1: $\langle 6, 3, 2, 4 \rangle$

Step 2: $\langle 3, 6, 2, 4 \rangle$

Step 3: $\langle 2, 3, 6, 4 \rangle$

Step 4: $\langle 2, 3, 4, 6 \rangle$

$\boxed{\textbf{Analyzing Algorithms}}$

Predict resource utilization

1. Memory (space complexity)

2. Running time (time complexity)

Remark: Really depends on the model of computation (sequential or parallel). We usually assume sequential.

Analyzing Algorithms – Continued

Running time: the number of primitive operations used to solve the problem.

Primitive operations: e.g., addition, multiplication, comparisons.

Running time: depends on problem instance, often we find an upper bound: F(input size)

Input size: rigorous definition given later.

1. **Sorting:** number of items to be sorted

2. **Multiplication:** number of bits, number of digits.

3. **Graphs:** number of vertices and edges.

Three Cases of Analysis

Best Case: constraints on the input, other than size, resulting in the fastest possible running time.

Worst Case: constraints on the input, other than size, resulting in the slowest possible running time.
Example. In the worst case *Quicksort* runs in $\Theta(n^2)$ time on an input of n keys.

Average Case: average running time over every possible type of input (usually involve probabilities of different types of input).
Example. In the average case *Quicksort* runs in $\Theta(n \log n)$ time on an input of n keys. All $n!$ inputs of n keys are considered equally likely.

Remark: All cases are relative to the algorithm under consideration.

Three Analyses of Insertion Sorting

Best Case: $A[1] \leq A[2] \leq A[3] \leq \cdots \leq A[n]$.

The number of comparisons needed is equal to

$$\underbrace{1+1+1+\cdots+1}_{n-1} = n-1 = \Theta(n).$$

Worst Case: $A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$.

The number of comparisons needed is equal to

$$1+2+\cdots+(n-1) = \frac{n(n-1)}{2} = \Theta(n^2).$$

Average Case: $\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely.

Big Oh

$f(n) = O(g(n))$: $g(n)$ is an *asymptotically upper bound* for $f(n)$.

$$O(g(n)) = \{f(n) : \exists\ c > 0 \text{ and } n_0 > 0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}.$$

Remark: "$f(n) = O(g(n))$" means that

$$f(n) \in O(g(n)).$$

Examples:

(1) $n^2 + 2n = O(n^2)$.

(2) $200n^2 - 100n = O(n^2)$.

(3) $n \log_2 n = O(n^2)$.

(4) $n^2 \log_2 n \neq O(n^2)$.

(5) $\forall a, b > 1$, $\log_a n = O(\log_b n)$.

Big Omega

$f(n) = \Omega(g(n))$: $g(n)$ is an *asymptotically lower bound* for $f(n)$.

$$\Omega(g(n)) = \{f(n) : \exists\ c > 0 \text{ and } n_0 > 0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}.$$

Examples:

(1) $200n^2 - 100n = \Omega(n^2) = \Omega(n) = \Omega(1)$.

(2) $n^2 = \Omega(n)$.

(3) $n^2 \neq \Omega(n^2 \log n)$.

(3) Does $f(n) = O(g(n))$ imply $g(n) = \Omega(f(n))$?

(4) Does $f(n) = \Omega(g(n))$ imply $g(n) = O(f(n))$?

$\boxed{\textbf{Big Theta}}$

$f(n) = \Theta(g(n))$**:** $g(n)$ is an *asymptotically tight bound* for $f(n)$.

$$\Theta(g(n)) = \{f(n) : \exists\ n_0 > 0,\ c_1 > 0 \text{ and } c_2 > 0 \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}.$$

Examples:

(1) $5n^2 - 2n + 5 = \Theta(n^2)$

(2) $5n^2 - 2n + 5 = \Theta(n^2 + \log n)$

Note that if

$$f(n) = \Theta(g(n)),$$

then

$$f(n) = \Omega(g(n)) \quad \text{and} \quad f(n) = O(g(n)).$$

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In the other direction, if

$$f(n) = \Omega(g(n)) \quad \text{and} \quad f(n) = O(g(n)),$$

then

$$f(n) = \Theta(g(n)).$$

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$$\boxed{f(n) = \Theta(g(n)) \quad \Leftrightarrow \quad f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))}$$

Some thoughts on Algorithm Design

- *Algorithm Design*, as taught in this class, is mainly about designing algorithms that have small big $O()$ running times.

- "All other things being equal", $O(n \log n)$ algorithms will run more quickly than $O(n^2)$ ones and $O(n)$ algorithms will beat $O(n \log n)$ ones.

- Being able to do good algorithm design lets you identify the *hard parts* of your problem and deal with them effectively.

- Too often, programmers try to solve problems using brute force techniques and end up with slow complicated code! A few hours of abstract thought devoted to algorithm design could have speeded up the solution substantially *and* simplified it.

Note: After algorithm design one can continue on to *Algorithm tuning* which would further concentrate on improving algorithms by cutting cut down on the *constants* in the big $O()$ bounds. This needs a good understanding of both algorithm design principles and efficient use of data structures. In this course we will not go further into algorithm tuning. For a good introduction, see chapter 9 in *Programming Pearls, 2nd ed* by Jon Bentley.