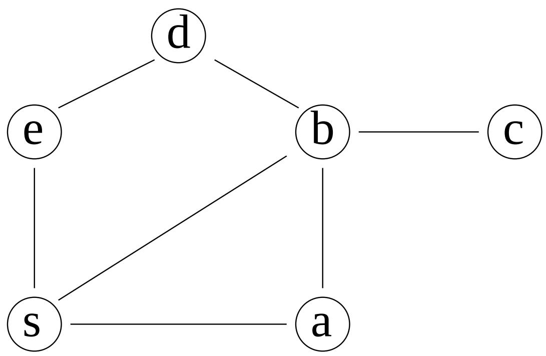
Lecture 6: Breadth-First Search

Outline of this Lecture

- The shortest path problem.
- The breadth-first search algorithm.
- The running time of BFS.
- The correctness proof.

Note: We introduce BFS for *undirected* graphs, but the same algorithm will also work for directed graphs.

Shortest Paths



Example: 3 simple paths from the source s to b:

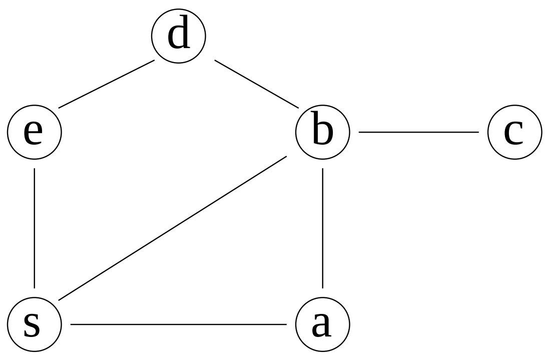
$$\langle s, b \rangle, \ \langle s, a, b \rangle, \ \langle s, e, d, b \rangle$$

of length 1, 2, 3 respectively. So the shortest path from s to b is $\langle s, b \rangle$. The shortest paths from s to other vertices are

$$\langle s, a \rangle, \ \langle s, b \rangle, \ \langle s, b, c \rangle, \ \langle s, b, d \rangle, \langle s, e \rangle.$$

There are two shortest paths from s to d.

The Shortest Path Problem



Distance $d[v]$: The length of a shortest path from s to v.
For example $d[c] = 2$. We define $d[s] = 0$.

The Problem: Given a graph $G = (V, E)$ and a *source* vertex $s \in V$, find the distances $d[v]$ and a shortest path from s to each other vertex in G.

What Does the Breadth-First Search Do

Given a graph $G = (V, E)$, the BFS returns

- the distances $d[v]$ from s to v;
- the predecessors $\mathtt{pred}[v]$, which is used to derive a shortest path from s to every other vertex v.

BFS is actually returning a *shortest path tree* in which the unique path from s to node u is a shortest path from s to u in the original graph.

Remarks: In addition to the two arrays $d[v]$ and $\mathtt{pred}[v]$, the BFS also uses another auxiliary array $color[v]$, which has three possible values:
• white (W, "undiscovered"),
• gray (G, "discovered" but not "processed"),
• black (B, "discovered" and "processed").

The Breath-First Search

The Idea of the BFS:

Visit the vertices as follows:
1. visit all vertices at distance 1
2. visit all vertices at distance 2
3. visit all vertices at distance 3
etc.

Initially, s is made gray.

When a gray vertex is visited, its color is changed to black, and the color of all white neighbors is changed to gray.

Gray vertices are kept in a queue Q.

The Breath-First Search

More details.

G given by its adjacency list.

Initialization, first part:

For each vertex $u \in V$,
$color[u] = W$; $d[u] = \infty$; pred$[u] = NIL$;

Initialization, second part:

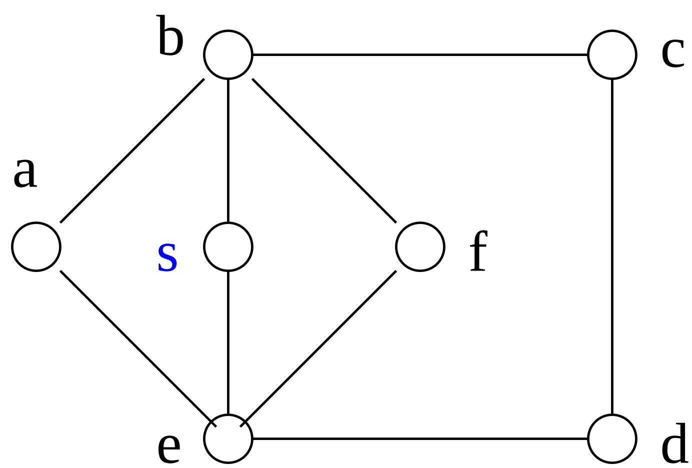
$color[s] = G$, $d[s] = 0$, $Q = \langle s \rangle$.

Main loop:

if Q is nonempty,
$u =$ dequeue(Q)
for each $v \in adj[u]$,
if $(color[v] == W)$, do
$color[v] = G$
$d[v] = d[u] + 1$
pred$[v] = u$
put v in Q
$color[u] = B$.

Example of the Breadth-First Search

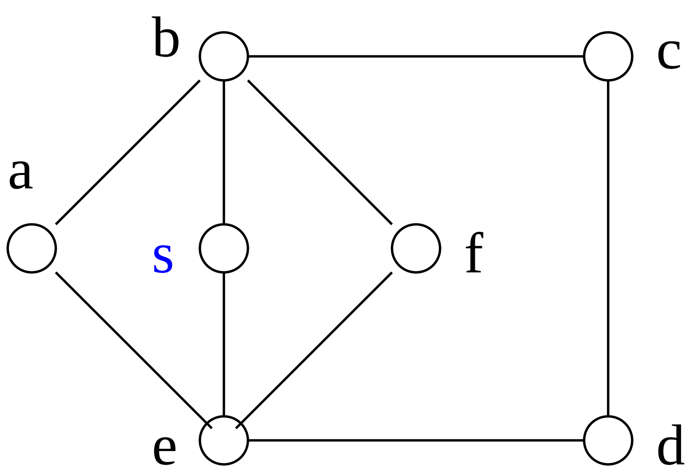
Problem: Given the following undirected graph and source vertex, find the distance from s to each vertex $u \in V$ and the predecessor $\text{pred}[u]$ along a shortest path by following the algorithm described earlier.



Example – Continued

Initialization, first part

vertex u	s	a	b	c	d	e	f
$color[u]$	W	W	W	W	W	W	W
$d[u]$	∞	∞	∞	∞	∞	∞	∞
pred$[u]$	NIL	NIL	NIL	NIL	NIL	NIL	NIL

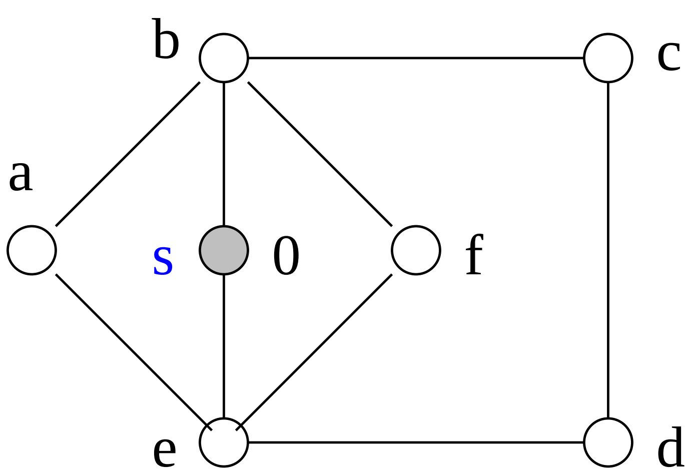


Example – Continued

Initialization, second part

vertex u	s	a	b	c	d	e	f
$color[u]$	G	W	W	W	W	W	W
$d[u]$	0	∞	∞	∞	∞	∞	∞
pred$[u]$	NIL	NIL	NIL	NIL	NIL	NIL	NIL

$Q = \langle s \rangle$
(Put s into Q (discovered) & mark "G", meaning "unprocessed")



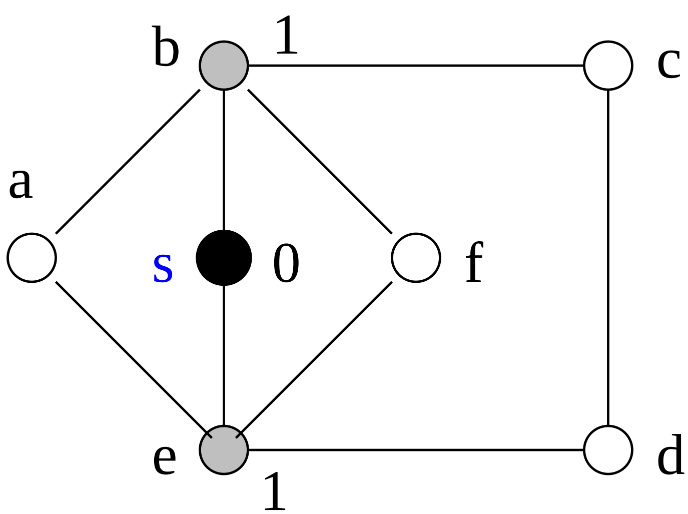
Example – Continued

While loop, first iteration

Dequeue s from Q. Find $adj[s] = \langle b, e \rangle$.
Mark b and e "G", mark s "B".
Update $d[b]$, $d[e]$, pred$[b]$, pred$[e]$.
Put b, e in Q.

vertex u	s	a	b	c	d	e	f
$color[u]$	B	W	G	W	W	G	W
$d[u]$	0	∞	1	∞	∞	1	∞
pred$[u]$	NIL	NIL	s	NIL	NIL	s	NIL

$Q = \langle b, e \rangle$



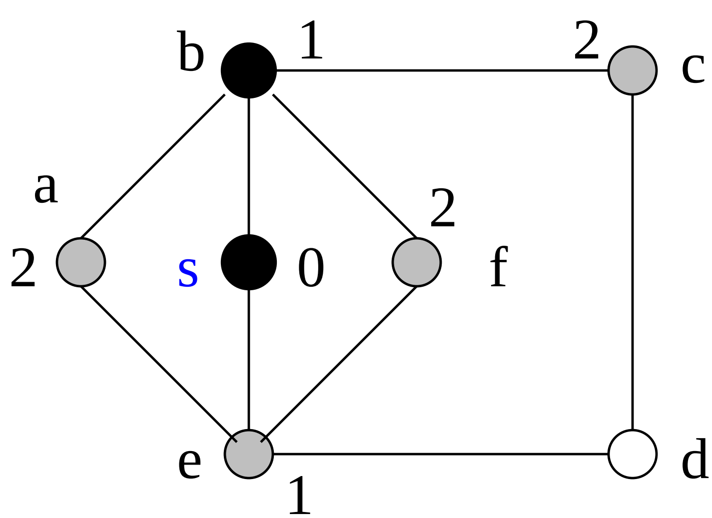
Example – Continued

While loop, second iteration

Dequeue b from Q. Find $adj[b] = \langle s, a, c, f \rangle$.
Mark a, c, f "G", mark b "B".
Update $d[a]$, $d[c]$, $d[f]$, pred$[a]$, pred$[d]$, pred$[f]$.
Put a, c, f in Q.

vertex u	s	a	b	c	d	e	f
$color[u]$	B	G	B	G	W	G	G
$d[u]$	0	2	1	2	∞	1	2
pred$[u]$	NIL	b	s	b	NIL	s	b

$Q = \langle e, a, c, f \rangle$



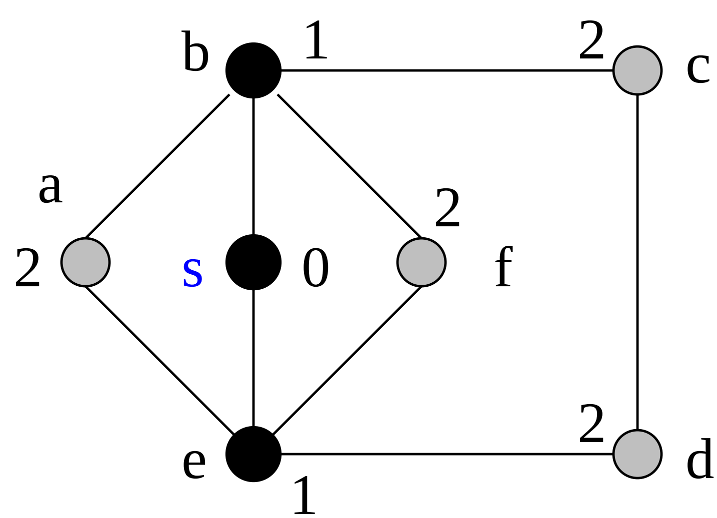
Example – Continued

While loop, third iteration

Dequeue e from Q. Find $adj[e] = \langle s, a, d, f \rangle$.
Mark d “gray”, mark e “B”.
Update $d[d]$, pred$[d]$.
Put d in Q.

vertex u	s	a	b	c	d	e	f
$color[u]$	B	G	B	G	G	B	G
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \langle a, c, f, d \rangle$



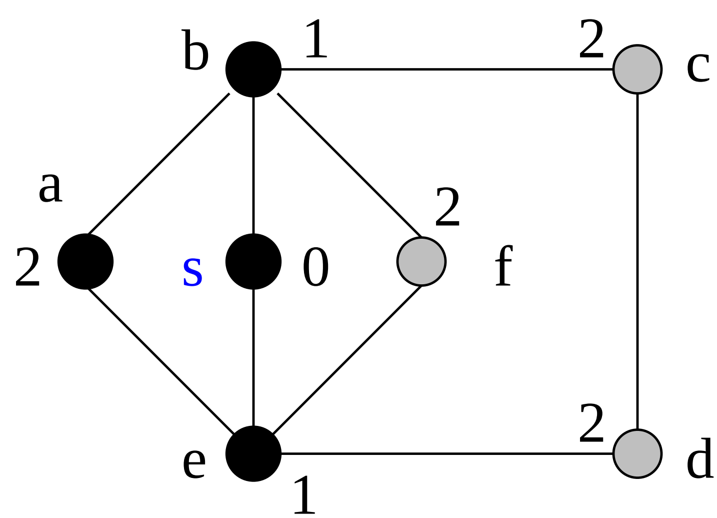
Example – Continued

While loop, forth iteration

Dequeue a from Q. Find $adj[a] = \langle b, e \rangle$.
Mark a "B".

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	G	G	B	G
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \langle c, f, d \rangle$



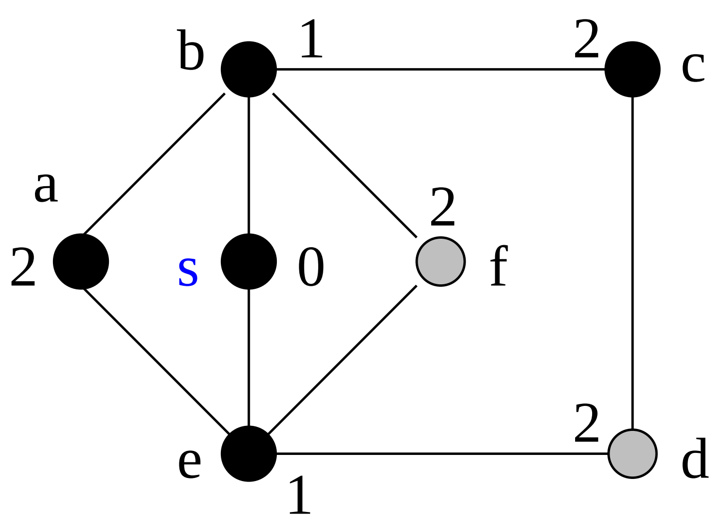
Example – Continued

While loop, fifth iteration

Dequeue c from Q. Find $adj[c] = \langle b, d \rangle$.
Mark c "B".

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	B	G	B	G
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \langle f, d \rangle$



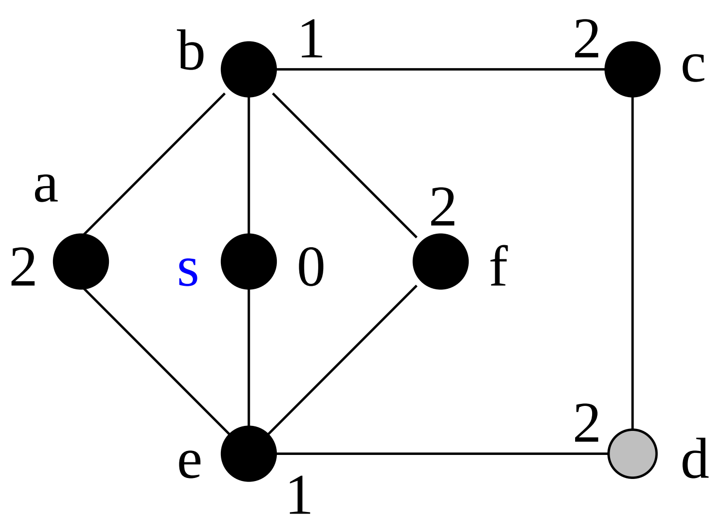
Example – Continued

While loop, sixth iteration

Dequeue f from Q. Find $adj[f] = \langle b, e \rangle$.
Mark f "B".

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	B	G	B	B
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \langle d \rangle$



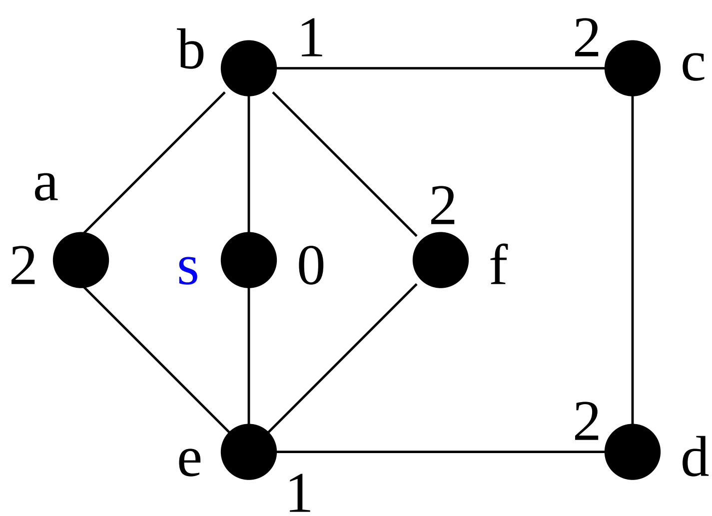
Example – Continued

While loop, seventh iteration

Dequeue d from Q. Find $adj[d] = \langle c, e \rangle$.
Mark d "B".

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	B	B	B	B
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \emptyset$



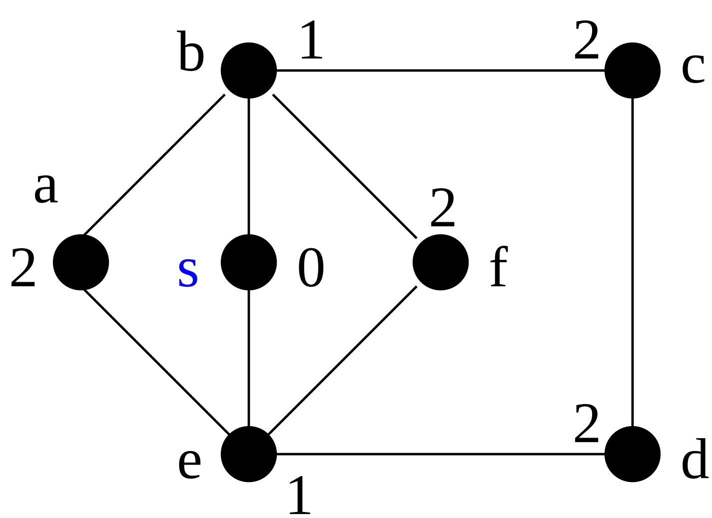
Example – Continued

While loop, eigth iteration

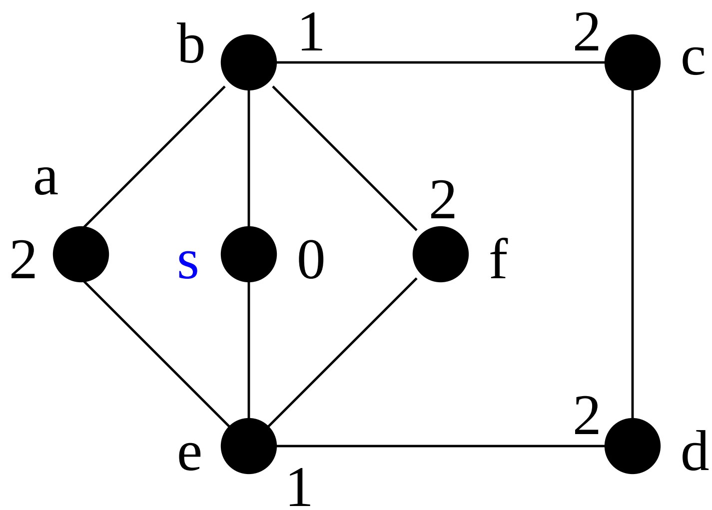
Since Q is empty, stop.

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	B	B	B	B
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

$Q = \emptyset$



Example – Continued



Question: How do you construct a shortest path from s to any vertex by using the following table?

vertex u	s	a	b	c	d	e	f
$color[u]$	B	B	B	B	B	B	B
$d[u]$	0	2	1	2	2	1	2
pred$[u]$	NIL	b	s	b	e	s	b

The Breadth-First Search Algorithm

```
for (each vertex u ∈ V)
{   color[u] = W;
    d[u] = ∞;
    pred[u] = NIL;
}
color[s] = G; d[s] = 0; enqueue(Q, s);
if (Q is nonempty)
{   u = dequeue(Q);
    for (each v ∈ adj[u])
        if (color[v] == W)
        {   color[v] = G;
            d[v] = d[u] + 1;
            pred[v] = u;
            enqueue(C, v);
        }
    color[u] = B;
}
```

Analysis of the Breadth-First Search Algorithm

Let $n = |V|$ and $e = |E|$. We assume that it takes one time unit to test the color of a vertex, or to update the color of a vertex, or to compute $d[v] = d[u] + 1$, or to set $\text{pred}[v] = u$, or to enqueue, or to dequeue.

The following analysis is valid for **connected** graphs.

- The initialization requires $3n + 3$ time units.

- Each vertex u must be processed and is processed once. What is the total amount of time for processing u?

  For each $v \in adj[u]$, if $color[v] = W$, the inner loop takes 4 time units. Otherwise the inner loop will not be carried out.

  $dequeue[u]$ and set $color[u] = B$ take 2 time units. Hence the total amount of time needed for processing u is at most

  $$5 \deg(u) + 2$$

  Hence the total amount of time needed for processing all the vertices is at most

  $$\sum_{u \in V} (5 \deg(u) + 2) = 10e + 2n.$$

- Hence

$$\begin{aligned} T(n, e) &\leq (3n + 3) + (10e + 2n) \\ &= 5n + 10e + 3 = O(n + e). \end{aligned}$$

Analysis of the Breadth-First Search Algorithm

The analysis can be improved:

Each vertex is colored G exactly once.

Therefore, the inner loop is executed exactly $(n-1)$ times.

Hence

$$\begin{aligned} T(n,e) &= (3n+3)+4(n-1)+\sum_{u\in V}(\deg(u)+2) \\ &= (3n+3)+(4n-4)+(2e+2n) \\ &= 9n+2e-1. \end{aligned}$$

Compare to

$$T(n,e) \leq 5n+10e+3.$$

Remark: Note that $e \leq n(n-1)/2$.

Since the graph is connected, $e \geq n-1$.

If $e = \Theta(n)$, then $T(n,e) = \Theta(n)$.

Graphs that are not connected

The BFS algorithm also works for graphs that are not connected. For such graphs, only the vertices v that are in the same component as s will get a value $d[v] \neq \infty$.

In particular, we can use the array $d[\,]$ at the end of the computation to decide if the graph is connected.

Alternatively, we can use the array $color[\,]$ or the array $pred[\,]$. Explain why.

How is the analysis of the BFS algorithm changed if we do not assume that the graph is connect?

We can actually modify BFS so that it returns a forest.
More specifically, if the original graph is composed of connected components $C_1, C_2, \ldots, C_k$ then BFS will return a tree corresponding to each C_i.

$BFS(s)$ Start BFS
$color[s] = G$; $d[s] = 0$; enqueue(Q, s);
if $(Q$ is nonempty$)$
$\{$ $u =$ dequeue(Q);
  for (each $v \in adj[u]$)
    if $(color[v] == W)$
    $\{$ $color[v] = G$;
      $d[v] = d[u] + 1$;
      pred$[v] = u$;
      enqueue(C, v);
    $\}$
  $color[u] = B$;
$\}$ End BFS
for (each vertex $u \in V$) Initialize
$\{$ $color[u] = W$;
  $d[u] = \infty$;
  pred$[u] = NIL$;
$\}$
for (each vertex $u \in V$) Start Connected Component
  if $d[u] \neq \infty)$
    $BFS(u)$;

Correctness of the BFS Algorithm

The correctness of the BFS algorithm consists of the following two parts.

1. Prove that the BFS algorithm outputs the correct distance $d[v]$.

2. Prove that the paths obtained by using the array $\text{pred}[v]$ are the shortest.

Since the path constructed with the array $\text{pred}[v]$ has length exactly $d[v]$, we need to prove only the first part!

Correctness of the BFS Algorithm

Observations: Any vertex v in Q has a real value $d[v] \neq \infty$.

For $u, v \in Q$ at any time, if $d[u] < d[v]$ then u was discovered earlier than v and (will be processed) earlier than v.

Proof: No proof is given here. You are encouraged to come up with your own proof.

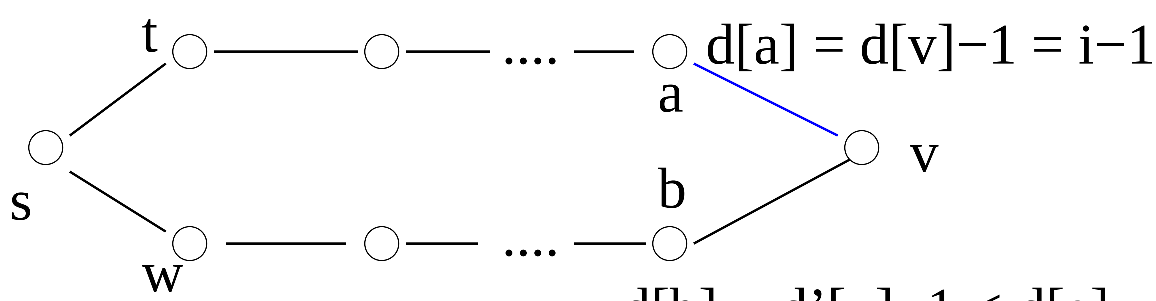
Theorem: The BFS algorithm outputs the correct distance $d[v]$.

Proof: See next page.

Correctness of the BFS Algorithm

Proof: By induction on $d[v]$. If $d[v] = 0$, then $v = s$. The conclusion is true.

Assume that $d[v]$ is the correct distance for all $d[v] < i$. Consider the case $d[v] = i$. If $d[v]$ were not the correct distance, then the true distance $d'[v] < d[v]$. We then have two paths:



d[b] = d'[v]−1 < d[a]

d'[v]−1 < d[v]−1 = i−1

(1) st...a must be a shortest path by induction hypothesis as d[a]=d[v]−1=i−1<i

(2) sw...b must be a shortest path as it is a subpath of the shortest path sw...bv

(3) a distinct from b because d[b]<d[a], while both d[a] and d[b] are true distance

Since $d[b] < d[a]$, b should be processed earlier than a, and should discover v. This is contrary to that a discovered v.