Lecture 8: DFS and Topological Sort

CLRS 22.3, 22.4

Outline of this Lecture

- Recalling Depth First Search.
- The time-stamp structure.
- Using the DFS for cycle detection.
- Topological sort with the DFS.

What does DFS Do?

Given a digraph $G = (V, E)$, DFS traverses all vertices of G and

- constructs a forest, together with a set of source vertices; and

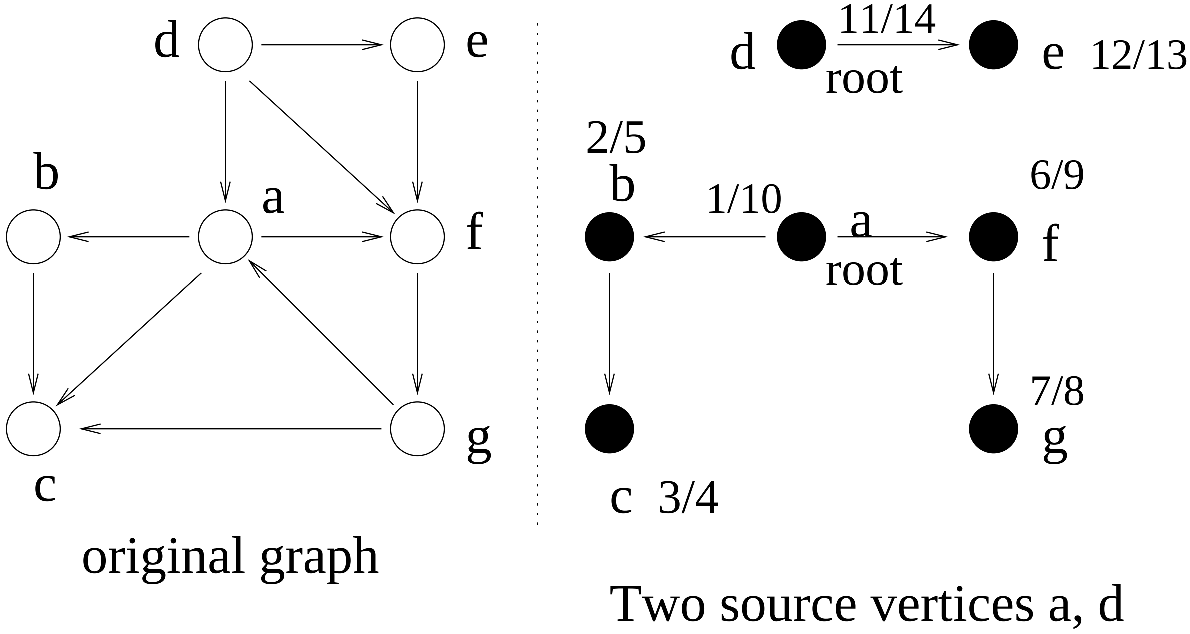
- outputs two time unit arrays, $d[v]/f[v]$.

DFS Forest: DFS constructs a forest $F = (V, E_f)$, a collection of trees, where

$E_f = \{(pred[v], v)|$ where DFS calls are made$\}$

A Depth First Search Example

Example



Question: What can we do with the returned data?

Classification of the Edges of a Digraph

Tree edges: those on the DFS forest.

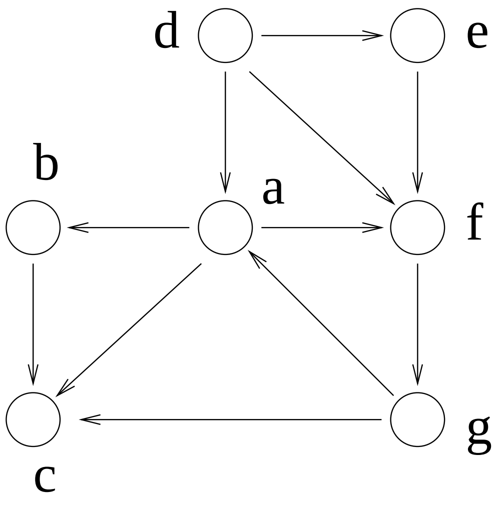
The remaining edges fall in three categories:

Forward edges: (u, v) where v is a proper descendent of u in the tree. [$u \neq v$.]

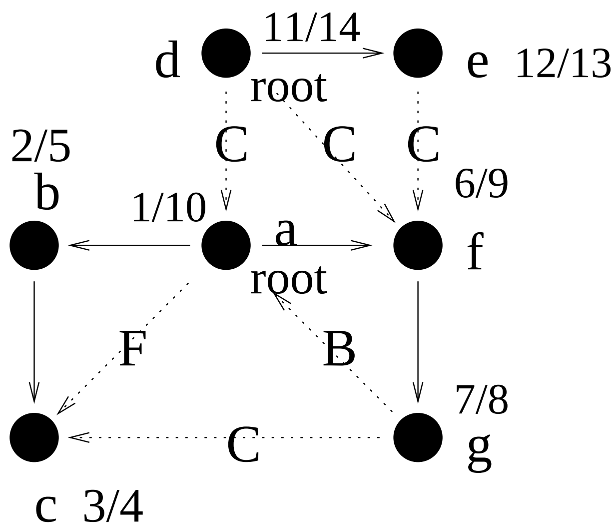
Back edges: (u, v), where vertex v is an ancestor of u in the tree. [$u = v$ is allowed.]

Cross edges: (u, v) where u and v are not ancestors or descendents of one another (in fact, the edges may go between different trees).

Example of the Classification of Edges



original graph



C: cross, F: forward,
B: back edge

Remark: Since the forest obtained with the DFS is not unique, the classification is not unique.

Time-Stamp Structure in DFS

There is also a nice structure to the time stamps, which is referred to as *Parenthesis Structure*.

Lemma 1 Given a digraph $G = (V, E)$, any DFS Forest for G, and any two vertices $u, v \in V$,

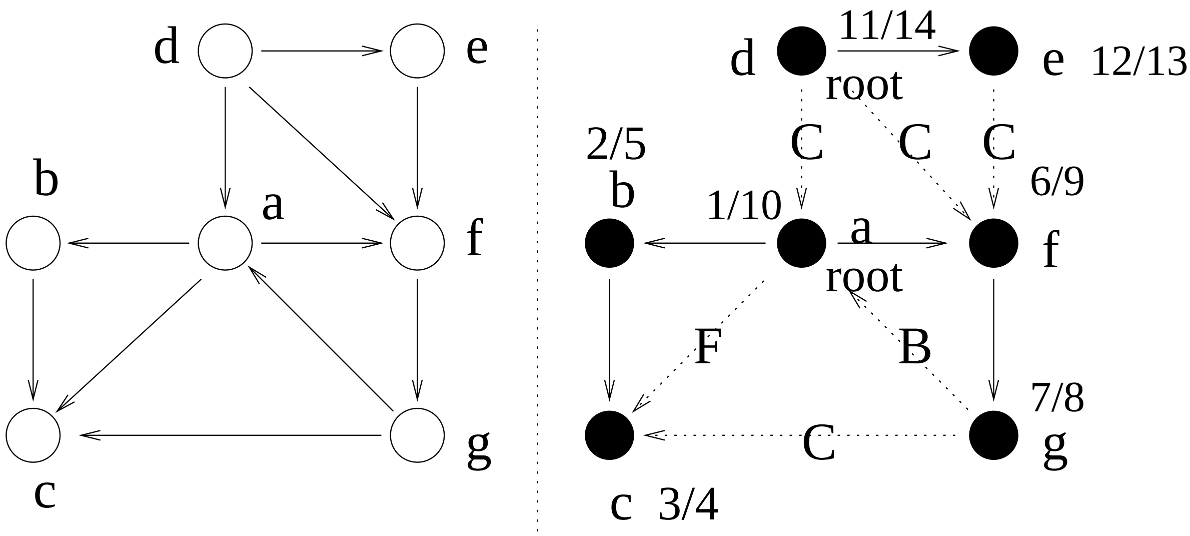
- u is a descendent of v *in the DFS forest* if and only if $[d[u], f[u]]$ is a subinterval of $[d[v], f[v]]$: $d[v] < d[u] < f[u] < f[v]$
- u is unrelated to v *in the DFS forest* if and only if $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint intervals: $f[u] < d[v]$ or $f[v] < d[u]$
- $d[u] < d[v] < f[u] < f[v]$ and $d[v] < d[u] < f[v] < f[u]$ are not possible

Cycles in digraphs: Applications of DFS

Claim: Given a digraph, DFS can be used to determine whether it contains any cycles.

Lemma 2: Given a digraph G, and any DFS tree of G, tree edges, forward edges, and cross edges all go from a vertex of higher finish time to a vertex of lower finish time. Back edges go from a vertex of lower finish time to a vertex of higher finish time.

Proof: The conclusions follow from Lemma 1.



original graph

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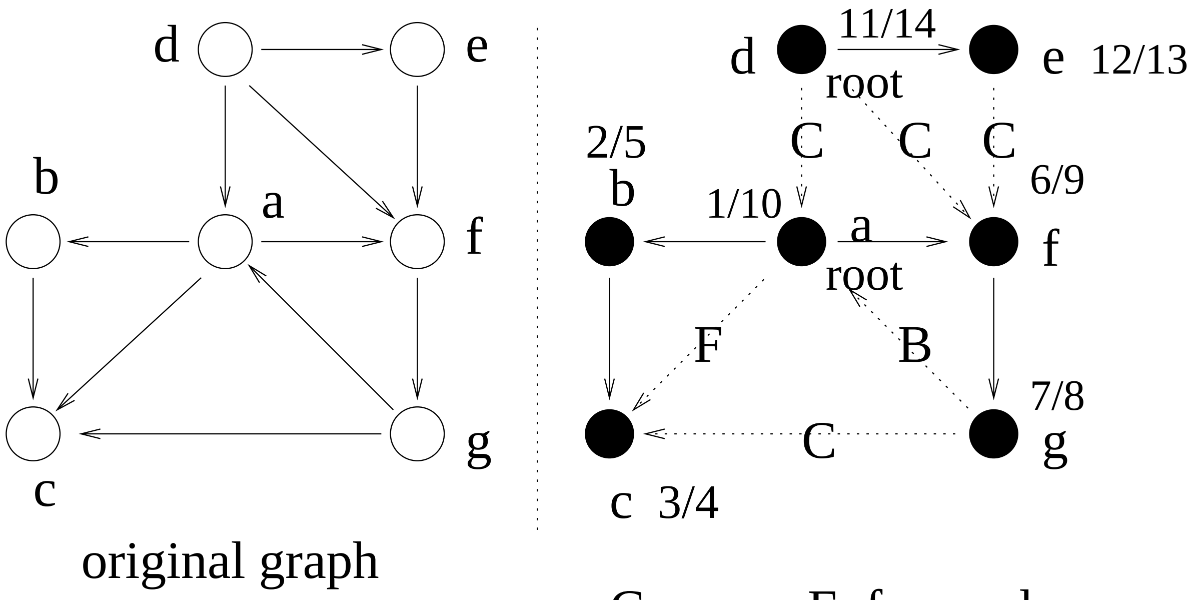
When Is a Digraph Acyclic

Lemma 3: A digraph is acyclic if and only if any DFS forest of G yields no back edges.

Example:



When Is a Digraph Acyclic

Lemma 3: A digraph is acyclic if and only if any DFS forest of G yields no back edges.

Proof of $\Leftarrow$: Suppose there are no back edges. By Lemma 2, all edges go from a vertex of higher finish time to a vertex of lower finish time. Hence there can be no cycles.

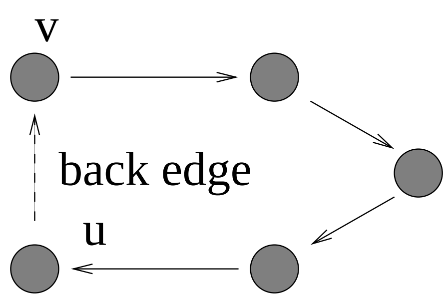
When Is a Digraph Acyclic

Lemma 3: A digraph is acyclic (a DAG) if and only if any DFS forest of G yields no back edges.

Proof of $\Rightarrow$: Assume that G has no cycles. We prove that G has no back edges by contradiction.

Suppose there is a back edge (u, v).
Then v is an ancestor of u in a rooted DFS tree.
There is a path $v \rightarrow ... \rightarrow u$ in the DFS tree.

The back edge + the path gives a cycle. A contradiction!



Cycle Detection with the DFS

Cycle detection: becomes back edge detection by Lemma 3!

Problem: Modify the DFS algorithm slightly to give an algorithm for cycle detection.
This can always be done by first running the algorithm and assigning the $d[v]$ and $f[v]$ values and then running through all of the edges one more time, seeing if any of them are back edges. This would take $O(n+e)$ time for the DFS and $O(e)$ time for the scan through all of the edges. In total, this uses $O(n+e)$ time.

How could you solve this problem *online* by identifying back edges while the DFS algorithm is still running?

Topological Sorting; graphs

If $G = (V, E)$ is a DAG then a topological sorting of V is a linear ordering of V such that for each edge (u, v) in the DAG, u appears before v in the linear ordering.

Example: Let $V = \{1, 3, 4, 5, 6, 12\}$ and have $(a, b) \in E$ if and only if $a|b$. This is partial order, but not a linear one.
There are several topological sortings of G (how many?), for example:

$\langle 1, 3, 4, 5, 6, 12\rangle$, $\langle 1, 4, 3, 6, 12, 5\rangle$, $\langle 1, 5, 3, 6, 4, 12\rangle$.

Topological Sorting; graphs

If $G = (V, E)$ is a DAG then a topological sorting of V is a linear ordering of V such that for each edge (u, v) in the DAG, u appears before v in the linear ordering.

Idea of Topological Sorting: Run the DFS on the DAG and output the vertices in reverse order of finishing time.

Correctness of the Idea: By lemma 2, for every edge (u, v) in a DAG, the finishing time of u is greater than that of v, as there are no back edges and the remaining three classes of edges have this property.

Topological Sort: the Algorithm

The Algorithm:

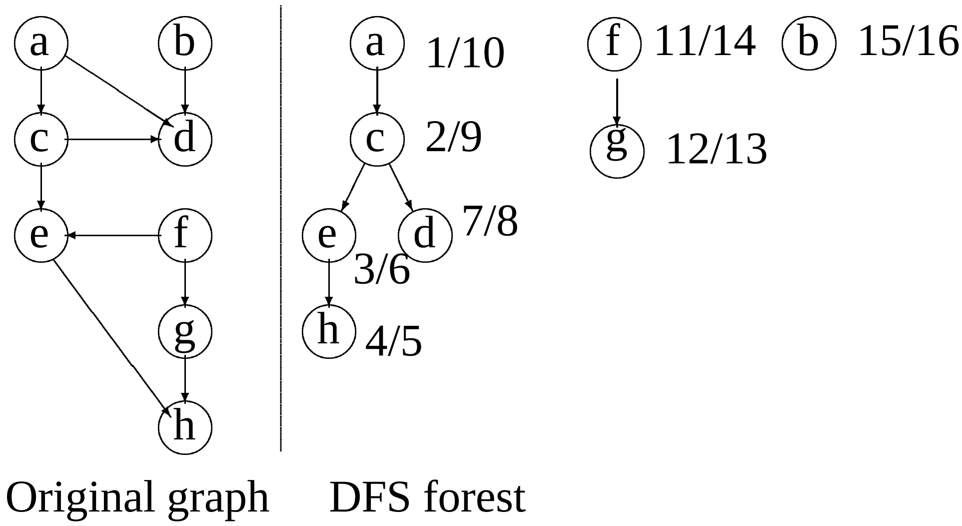
1. Run DFS(G), computing finish time $f[v]$ for each vertex v;

2. As each vertex is finished, insert it onto the front of a list;

3. Output the list

Running time: $\Theta(n + e)$, the same as DFS.

Topological Sort: Example



Final order: $\langle b, f, g, a, c, d, e, h \rangle$.