Lecture 15: The Floyd-Warshall Algorithm

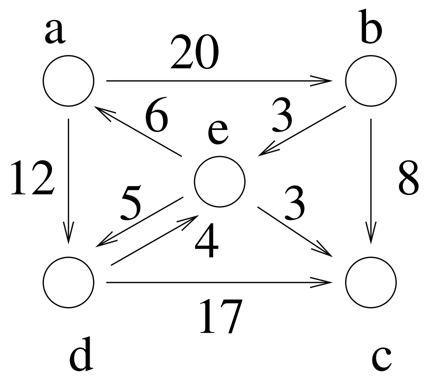
CLRS section 25.2

Outline of this Lecture

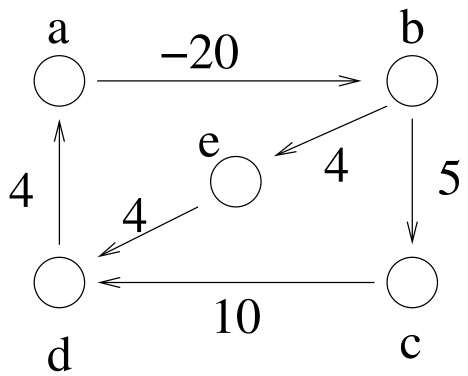
- Recalling the all-pairs shortest path problem.
- Recalling the previous two solutions.
- The Floyd-Warshall Algorithm.

The All-Pairs Shortest Paths Problem

Given a weighted digraph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}$, where R is the set of real numbers, determine the length of the shortest path (i.e., distance) between all pairs of vertices in G. Here we assume that there are no cycle with zero or negative cost.



without negative cost cycle



with negative cost cycle

Solutions Covered in the Previous Lecture

Solution 1: Assume no negative *edges*.

Run Dijkstra's algorithm, n times, once with each vertex as source.

$O(n^3 \log n)$. $O(n^3)$ with more sophisticated data structures.

Solution 2: Assume no negative *cycles*.

Dynamic programming solution, based on a natural decomposition of the problem.

$O(n^4)$. $O(n^3 \log n)$ using " repeated squaring".

This lecture: Assume no negative *cycles*.
develop another dynamic programming algorithm, the *Floyd-Warshall algorithm,* with time complexity $O(n^3)$.
Also illustrates that there can be more than one way of developing a dynamic programming algorithm.

$\boxed{\textbf{Solution 3: the Input and Output Format}}$

As in the previous dynamic programming algorithm, we assume that the graph is represented by an $n \times n$ matrix with the weights of the edges:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ w(i,j) & \text{if } i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

Output Format: an $n \times n$ distance $D = [d_{ij}]$ where d_{ij} is the distance from vertex i to j.

Step 1: The Floyd-Warshall Decomposition

Definition: The vertices $v_2, v_3, ..., v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, ..., v_l \rangle$.

- Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that *all* intermediate vertices on the path (if any) are in set $\{1, 2, \ldots, k\}$.

  $d_{ij}^{(0)}$ is set to be w_{ij}, i.e., no intermediate vertex.
  Let $D^{(k)}$ be the $n \times n$ matrix $[d_{ij}^{(k)}]$.

- Claim: $d_{ij}^{(n)}$ is the distance from i to j. So our aim is to compute $D^{(n)}$.

- Subproblems: compute $D^{(k)}$ for $k = 0, 1, \cdots, n$.

Step 2: Structure of shortest paths

Observation 1:
A shortest path does not contain the same vertex twice.
Proof: A path containing the same vertex twice contains a cycle. Removing cycle gives a shorter path.

Observation 2: For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1, 2, \ldots, k\}$, there are two possibilities:

1. k is not a vertex on the path,
The shortest such path has length $d_{ij}^{(k-1)}$.

2. k is a vertex on the path.
The shortest such path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Step 2: Structure of shortest paths

Consider a shortest path from i to j containing the vertex k. It consists of a subpath from i to k and a subpath from k to j.
Each subpath can only contain intermediate vertices in $\{1, ..., k-1\}$, and must be as short as possible, namely they have lengths $d_{ik}^{(k-1)}$ and $d_{kj}^{(k-1)}$.

Hence the path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Combining the two cases we get

$$d_{ij}^{(k)} = \min\left\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\}.$$

Step 3: the Bottom-up Computation

- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, ..., n$.

The Floyd-Warshall Algorithm: Version 1

Floyd-Warshall(w, n)

```
{ for i = 1 to n do                         initialize
    for j = 1 to n do
    { D^0[i, j] = w[i, j];
      pred[i, j] = nil;
    }

  for k = 1 to n do                         dynamic programming
    for i = 1 to n do
      for j = 1 to n do
        if (d^(k-1)[i, k] + d^(k-1)[k, j] < d^(k-1)[i, j])
            {d^(k)[i, j] = d^(k-1)[i, k] + d^(k-1)[k, j];
             pred[i, j] = k;}
        else d^(k)[i, j] = d^(k-1)[i, j];
  return d^(n)[1..n, 1..n];
}
```

Comments on the Floyd-Warshall Algorithm

- The algorithm's running time is clearly $\Theta(n^3)$.

- The predecessor pointer $\texttt{pred[i, j]}$ can be used to extract the final path (see later).

- Problem: the algorithm uses $\Theta(n^3)$ space.
  It is possible to reduce this down to $\Theta(n^2)$ space by keeping only one matrix instead of n.
  Algorithm is on next page. Convince yourself that it works.

The Floyd-Warshall Algorithm: Version 2

Floyd-Warshall(w, n)

$\{$ for $i = 1$ to n do   initialize

    for $j = 1$ to n do

    $\{$ $d[i, j] = w[i, j]$;

      $pred[i, j] = nil$;

    $\}$

  for $k = 1$ to n do   dynamic programming

    for $i = 1$ to n do

      for $j = 1$ to n do

        if $(d[i, k] + d[k, j] < d[i, j])$

            $\{d[i, j] = d[i, k] + d[k, j]$;

            $pred[i, j] = k;\}$

  return $d[1..n, 1..n]$;

$\}$

Extracting the Shortest Paths

The predecessor pointers `pred[i, j]` can be used to extract the final path. The idea is as follows.

Whenever we discover that the shortest path from i to j passes through an intermediate vertex k, we set $pred[i, j] = k$.

If the shortest path does not pass through any intermediate vertex, then $pred[i, j] = nil$.

To find the shortest path from i to j, we consult $pred[i, j]$. If it is nil, then the shortest path is just the edge (i, j). Otherwise, we recursively compute the shortest path from i to $pred[i, j]$ and the shortest path from $pred[i, j]$ to j.

The Algorithm for Extracting the Shortest Paths

```
Path(i, j)
{
    if (pred[i, j] = nil)    single edge
        output (i, j);
    else           compute the two parts of the path
    {
        Path(i, pred[i, j]);
        Path(pred[i, j], j);
    }
}
```

Example of Extracting the Shortest Paths

Find the shortest path from vertex 2 to vertex 3.

2..3	Path$(2,3)$	$pred[2,3] = 4$	
2..4..3	Path$(2,4)$	$pred[2,4] = 5$	
2..5..4..3	Path$(2,5)$	$pred[2,5] = nil$	$Output$(2,5)
25..4..3	Path$(5,4)$	$pred[5,4] = nil$	$Output$(5,4)
254..3	Path$(4,3)$	$pred[4,3] = 6$	
254..6..3	Path$(4,6)$	$pred[4,6] = nil$	$Output$(4,6)
2546..3	Path$(6,3)$	$pred[6,3] = nil$	$Output$(6,3)
25463			