Lecture 20: NP-Completeness II

Main Topics of this Lecture

1. Hamiltonian Cycle Problem.
   CLRS pp.1008-1012
2. The traveling salesman problem.
   CLRS pp 1012-1013
3. Some other $\mathcal{NPC}$ problems.

Recall how to prove $L \in \mathcal{NPC}$

To prove that $L \in \mathcal{NPC}$:

1. Prove $L \in \mathcal{NP}$.
2. Prove that $L' \leq_P L$ for some $L' \in \mathcal{NPC}$.

In this lecture, we will consider two problems involving Hamiltonian cycles.

Hamiltonian Cycle Problem

DHamCyc:

Given an undirected graph $G = (V, E)$,
is there a Hamiltonian cycle in G?
see Lecture 18 for definition of Hamiltonian Cycle.

Theorem: DHamCyc $\in \mathcal{NPC}$.

Proof: In lecture 18 we already proved that DHamCyc $\in \mathcal{NP}$.Using a complicated ‘gadget’ we can construct a polynomial reduction proving that

$$\text{DVC} \leq_{\text{P}} \text{DHamCyc}$$

and we have already seen that DVC (Decision Vertex Cover) is $\mathcal{NPC}$. Combining these two facts proved the theorem.

See CLRS P. 1008 for details of the reduction. For the purposes of this course you only need to know that the reduction exists but do not need to know the details.

Traveling Salesman Problem

Informally, the Traveling Salesman Problem (TSP) can be described as follows:

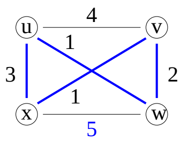
A salesman wishes to make a tour visiting each of n cities exactly once and finishing at the city he starts from.

There is an integer cost $c(i, j)$ to travel from city i to city j (or vise versa), and the salesman wishes to make the tour whose total cost is minimum, where the total cost is the sum of the individual costs along the edges of the tour.

Find the tour with the minimum cost.

Traveling Salesman Problem (TSP)

The TSP can be formulated as a graph problem:
Given a complete weighted undirected graph with n vertices, find a Hamiltonian cycle of least weight.



Tour (Hamiltonian cycle)
with minimum cost 7

Remark: The total number of Hamiltonian cycles is $n!$ which is **not** polynomial.

Decision TSP (DTSP)

Given a complete weighted undirected graph with n vertices, and a bound B, is there a Hamiltonian cycle of weight $\le B$?

Theorem: DTSP $\in \mathcal{NPC}$.

We will prove this theorem using polynomial-time reduction; we will show that
DTSP $\in \mathcal{NP}$ and DHamCyc $\le_P$ DTSP.

Proof that DTSP $\in \mathcal{NP}$

Certificate Verification:
Given a Hamiltonian cycle (certificate), we can, in time $O(n)$,

- check that the cycle contains each vertex once;
- check that the total weight $\le B$.

Hence DTSP $\in \mathcal{NP}$.

Proof that DHamCyc $\le_P$ DTSP

Steps for a Polynomial-Time Reduction:

(1) Define a transformation f mapping inputs to DHamCyc into inputs for DTSP.

(2) Show that f can be computed in polynomial-time.

(3) Prove that f is a reduction, that is, show that it transforms yes-inputs for DHamCyc into yes-inputs for DTSP, and no-inputs for DHamCyc into no-inputs for DTSP.

Proof that DHamCyc $\leq_P$ DTSP (cont)

Input to DHamCyc:
an undirected graph $G = (V, E)$.

Input to DTSP: an undirected complete graph G', a weight function c, and a bound B.

Step 1: Define a f : DHamCyc $\rightarrow$ DTSP by $f(G) = (G', c, B)$ where

$$G' = (V, \{\{u, v\} \mid u, v \in V,\ u \neq v\}),$$

$$c(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

$$B = |V|.$$

Step 2:
f is computed in time $O(|V|^2)$ and so is polynomial.

Proof that DHamCyc $\leq_P$ DTSP (cont)

Step 1, example.
Input to DHamCyc:
an undirected graph $G = (V, E)$.
Input to DTSP: an undirected complete graph G', a weight function c, and a bound B.

Mapping f:

Proof that DHamCyc $\leq_P$ DTSP (cont)

Step 3: Prove that f is a reduction.

The proof is done in two parts:

(3.a) If G has a Hamiltonian cycle,
then G' has a Hamiltonian cycle of weight at most n.

(3.b) If G' has a Hamiltonian cycle of weight at most n,
then G has a Hamiltonian cycle.

Proof that DHamCyc $\leq_P$ DTSP (cont)

(3.a) If G has a Hamiltonian cycle, then G' has a Hamiltonian cycle of weight at most n.

Proof: Suppose $s = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a Hamiltonian cycle of G

Then $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, n-1$ and $\{v_n, v_1\} \in E$.

Hence $c(v_i, v_{i+1}) = 1$ for $i = 1, 2, \ldots, n-1$ and $c(v_n, v_1) = 1$.

Therefore, s is a Hamiltonian cycle of weight n in G'.

Proof that DHamCyc $\leq_P$ DTSP (cont)

(3.b) If G' has a Hamiltonian cycle of weight at most n, then G has a Hamiltonian cycle.

Proof: Suppose $s = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$ is a Hamiltonian cycle of G' of weight at most n.

Since the cycle has n edges and each edge has weight at least 1, the total weight must be exactly n and each edge must have weight 1. Therefore, all the edges belong to E.

Hence s is a Hamiltonian cycle of G.

Combining (3.a) and (3.b) proves that f is a reduction. Hence DHamCyc $\leq_P$ DTSP.

A List of $\mathcal{NPC}$ Problems Covered

Satisfiability of Boolean formulas (SAT)
Decision clique (DCLIQUE)
Decision vertex cover (DVC)
Decision Hamiltonian cycle (DHamCyc)
Decision traveling salesman (DTSP)
Decision independent set (DIS)

Remark: Several thousand decision problems have been shown to be in $\mathcal{NPC}$.

A Review on $\mathcal{NP}$-Completeness

Input size of problems.

Polynomial-time and nonpolynomial-time algorithms.

Polynomial-time solvable problems.

Decision problems.

Optimization problems and their decision problems.

The classes $\mathcal{P}$, $\mathcal{NP}$, co-$\mathcal{NP}$, and $\mathcal{NPC}$.

Polynomial-time reduction.

How to prove $L \in \mathcal{P}$, or $\mathcal{NP}$, or $\mathcal{NPC}$?

Examples of problems in the classes.