1.2-1

Insertion sort: \( f(n) = 8n^2 \) steps. Merge sort: \( g(n) = 64n \log n \) steps

Solving the inequality \( 8n^2 \leq 64n \log n, \forall n \geq 0 \) gives \( n \leq 8 \log n \). One way to find \( n \) is by an approximation method such as fixed-point iteration. By fixed-point iteration, when \( n = 43 \), \( 8 \log n = 43.67 \). Now, substitute \( n = 43 \) and \( n = 44 \), we have:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
<th>( g(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>14792</td>
<td>14934</td>
</tr>
<tr>
<td>44</td>
<td>15548</td>
<td>15374</td>
</tr>
</tbody>
</table>

Both \( f(n) \) and \( g(n) \) are monotonically increasing. However, when \( n = 1 \), \( f(1) = 8 > g(1) = 0 \), Thus, for \( 1 < n < 44 \), the insertion sort beats the merge sort.

1-1

First express each function in term of \( t \) (i.e. \( f(t)=n \)):

1) \( \log n = t \Rightarrow n = \left[ 2^t \right] \)
2) \( \sqrt[n]{n} = t \Rightarrow n = \left[ t^2 \right] \)
3) \( n = t \)
4) \( n \log n = t \Rightarrow \log n = t/n \Rightarrow n = 2^{(t/n)} \)
5) \( n^2 = t \Rightarrow n = \left[ \sqrt{t} \right] \)
6) \( n^3 = t \Rightarrow n = \left[ 3^{1/3} t \right] \)
7) \( 2^n = t \Rightarrow \log 2^n = \log t \Rightarrow n = \left[ \log t \right] \)
8) \( n! = t \Rightarrow \) by guessing!

Then express each time unit in term of microseconds, where \( 1 s = 1 \times 10^6 \mu s \):

<table>
<thead>
<tr>
<th>Time Unit</th>
<th>1 s</th>
<th>1 min</th>
<th>1 hr</th>
<th>1 day</th>
<th>1 mo</th>
<th>1 yr</th>
<th>1 century</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E6=10^6</td>
<td>6E7</td>
<td>3.6E9</td>
<td>8.64E10</td>
<td>2.592E12</td>
<td>3.1104E13</td>
<td>3.1104E15</td>
<td></td>
</tr>
</tbody>
</table>

Now find \( n \) by substituting each time unit from the second table into the functions in the first table. Only a portion of the table is given. The rest is left to the reader as exercise:

<table>
<thead>
<tr>
<th>Time Unit</th>
<th>1 s</th>
<th>1 min</th>
<th>1 hr</th>
<th>1 day</th>
<th>1 mo</th>
<th>1 yr</th>
<th>1 century</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>( 2^{10^6} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2)</td>
<td>1E12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3)</td>
<td>1E6</td>
<td>6E7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4)</td>
<td>62746</td>
<td>2801417</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5)</td>
<td>1000</td>
<td>7745</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6)</td>
<td>100</td>
<td>391</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7)</td>
<td>19</td>
<td>25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8)</td>
<td>9</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.2-1

\( \Theta(n^3) \). [Reason: there exists a constant \( c1=1/10000, c2=1, n0=120000 \) such that \( 0 \leq c1 \cdot n^3 \leq f(n) \leq c2 \cdot n^3, \forall n \geq n0. \)]

2.3

a)

This problem is expected to be done using the way similar to p24 of the textbook. The analysis will be much easier if the while-loop is converted into a for-loop. Following the example, we have:
2

Thus, \( T(n) = c_1 + c_2(n+1) + c_3(n+1) = \Theta(n) \), where \( c_i > 0 \).

\( b) \)

1. \( y \leftarrow 0 \)
2. for \( i \leftarrow 0 \) to \( n \)
3. \hspace{1em} product \leftarrow 1
4. \hspace{1em} for \( j \leftarrow 1 \) to \( i \)
5. \hspace{2em} product \leftarrow product * x
6. \hspace{1em} y \leftarrow y + a[i] * product
7. return \( y \)

\begin{tabular}{|c|c|c|}
\hline
Step & Cost & Times \\
\hline
1 & \( c_1 \) & 1 \\
2 & \( c_2 \) & \( 1 + \sum_{i=0}^{n} 1 = n+2 \) \\
3 & \( c_3 \) & \( \sum_{i=0}^{n} 1 \) \\
4 & \( c_4 \) & \( \sum_{i=0}^{n} (\sum_{j=1}^{i} 1 + 1) = \sum_{i=0}^{n} (i + 1) = 1 + \sum_{i=1}^{n} (i + 1) = n(n-1) / 2 + n + 1 \) \\
5 & \( c_5 \) & \( \sum_{i=0}^{n} \sum_{j=1}^{i} 1 = n(n-1) / 2 \) \\
6 & \( c_6 \) & \( n+1 \) \\
7 & \( c_7 \) & 1 \\
\hline
\end{tabular}

Thus,
\[
T(n) = c_1 + c_2(n+2) + c_3(n+1) + c_4((n-1)/2 + n + 1) + c_5(n(n-1)/2 + c_6)n + 1) + c_7 = \Theta(n^2)
\]
3.1-1

Let \( f(n) \) and \( g(n) \) be any asymptotically nonnegative functions.
Then obviously, \( \exists n_0 \) such that \( \forall n \geq n_0 \), \( f(n) \) and \( g(n) \) are nonnegative, \( \forall n \geq n_0 \) \( f(n) \) and \( g(n) \) are nonnegative, \( \forall n \geq n_0 \) \( f(n) \) and \( g(n) \) are nonnegative, \( \forall n \geq n_0 \) \( f(n) \) and \( g(n) \) are nonnegative, \( \forall n \geq n_0 \) \( f(n) \) and \( g(n) \) are nonnegative\

By the definition of \( \Theta \)-notation, \( \Theta(f(n)+g(n)) = \{ h(n) : \text{there exists positive constants } c_1, c_2, n_0 \text{ such that } \forall n \geq n_0, 0 \leq c_1 (f(n)+g(n)) \leq h(n) \leq c_2 (f(n)+g(n)) \} \).

By (1), \( f(n) \leq f(n) + g(n) \) and \( g(n) \leq f(n) + g(n) \).
Now let \( c_2 = 1 \). Then,

\[
\max(f(n), g(n)) \leq c_2 (f(n) + g(n))
\]

\[
\Rightarrow 2 \max(f(n), g(n)) \leq 2(f(n) + g(n))
\]

\[
\Rightarrow f(n) + g(n) \leq 2 \max(f(n), g(n)) \leq 2(f(n) + g(n)) \quad \text{... (3)}
\]

\[
\Rightarrow \frac{1}{2} (f(n) + g(n)) \leq \max(f(n), g(n)) \leq (f(n) + g(n))
\]

By (2) and (3), \( 0 \leq c_1 (f(n)+g(n)) \leq \max(f(n),g(n)) \leq c_2 (f(n)+g(n)) \), \( \forall n \geq n_0 \) where \( c_1 = \frac{1}{2} \) and \( c_2 = 1 \).
\( \therefore \max(f(n), g(n)) = \Theta(f(n) + g(n)) \).

3.1-2

Let \( f(n) = (n+a)^b \), for real constants \( a, b > 0 \).

Case 1: \( a \geq 0 \).

Then \( 0 \leq n \leq n + a \leq 2n \) for all \( n \). Also, \( \log(n) \leq \log(n+a) \leq \log(2n) \).

Since \( b > 0 \), \( a^b \) is monotonically increasing in \( b \).

\[
\log n \leq b \log(n+1) \leq b \log(2n)
\]

\[
\Rightarrow \log n^b \leq \log(n+a)^b \leq \log(2n)^b
\]

\[
\Rightarrow 0 \leq n^b \leq (n+a)^b \leq (2^n)^b
\]

\( \therefore \exists c_1 = 1, c_2 = 2^b, n0 = a : 0 \leq c1n^b \leq (n+a)^b \leq c2n^b, \forall n \geq n0. \)

\( \therefore (n+a)^b = \Theta(n^b) \)

Case 2: \( a < 0 \) implies \( n + a = n - |a| \).

Then \( 0 \leq n/2 \leq n - |a| \leq n \), for all \( 2|a| \leq n \). Similar to case 1,

\[
0 \leq n/2 \leq n - |a| \leq n
\]

\[
\Rightarrow \log(n/2) \leq \log(n-|a|) \leq \log n
\]

\[
\Rightarrow \log(n/2)^b \leq \log(n-|a|)^b \leq \log n^b
\]

\[
\Rightarrow 0 \leq (1/2)^b n^b \leq (n-|a|)^b \leq n^b
\]

\( \therefore \exists c1 = (1/2)^b, c2 = 1. \)

\( \therefore (n - |a|)^b = \Theta(n^b) \)

Combining Case 1 + case 2, \( (n+a)^b = \Theta(n^b) \).