Wing Kuen Lee <cswkl@ust.hk>

4.1-1

Assume T(k) ≤ c1 lg (k – c2), for k < n

$$
\begin{aligned}
T(n) &\le c_1 \lg(\lceil n/2 \rceil - c_2) + 1 \\
&\le c_1 \lg(\tfrac{n+1}{2} - c_2) + 1 \\
&= c_1 \lg((n+1-2c_2)/2) + 1 \\
&= c_1 \lg(n+1-2c_2) - c_1 + 1 \\
&= c_1(\lg(n - c_2 - (c_c - 1)) - (c_1 - 1) \\
&\le c_1 \lg(n - c_2), \text{if } c_2 \ge 1, c_1 \ge 1
\end{aligned}
$$

Thus, the solution of T(n) is O(lg *n*).

4.1-2

$$
\text{Assume } T(k) \ge 2c \cdot n \lg n
$$

$$
\begin{aligned}
T(n) &\ge 2(2c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor) + n \\
&\ge 4c(\tfrac{n-1}{2}) \lg(\tfrac{n-1}{2}) + n \\
&\ge 4c(n/4)\lg(n/4) + n, \text{for } n \ge 2 \\
&= c \cdot n \lg n - 2cn + 2 \\
&= c \cdot n \lg n + n(1 - 2c) \\
&\ge c \cdot n \lg n, \text{for } c \le 1/4
\end{aligned}
$$

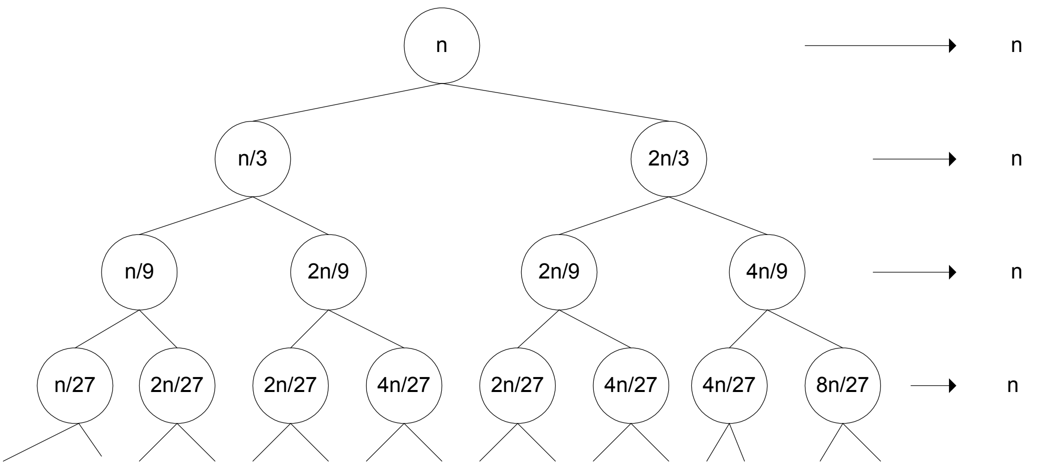
∴ T(*n*)∈Ω(*n* lg n).
Page 64 of the textbook shows that T(*n*)∈O(*n* lg n).
Therefore, T(*n*)∈Θ(*n* lg n).

4.2-1

$$
\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n = n + 3T(\lfloor n/2 \rfloor) \\
&= n + 3(\lfloor n/2 \rfloor + 3T(\lfloor n/4 \rfloor)) = n + 3(\lfloor n/2 \rfloor + 3(\lfloor n/4 \rfloor + 3T(\lfloor n/8 \rfloor))) \\
&\le n + 3n/2 + 9n/4 + 27(n/8) + \ldots + 3^{\lg n}(n/2^{\lg n}) \\
&\le n(1 + 3/2 + 9/4 + 27/8 + \ldots) + \Theta(n^{\lg 3}) \\
&= n \sum_{i=0}^{\lg n - 1} (3/2)^i + \Theta(n^{\lg 3}) \\
&= n\left(\frac{(3/2)^{\lg n} - 1}{3/2 - 1}\right) + \Theta(n^{\lg 3}) = n\left(\frac{n^{\lg(3/2)} - 1}{1/2}\right) + \Theta(n^{\lg 3}) = n\left(\frac{n^{\lg 3 - 1} - 1}{1/2}\right) + \Theta(n^{\lg 3}) \\
&= O(n^{\lg 3})
\end{aligned}
$$

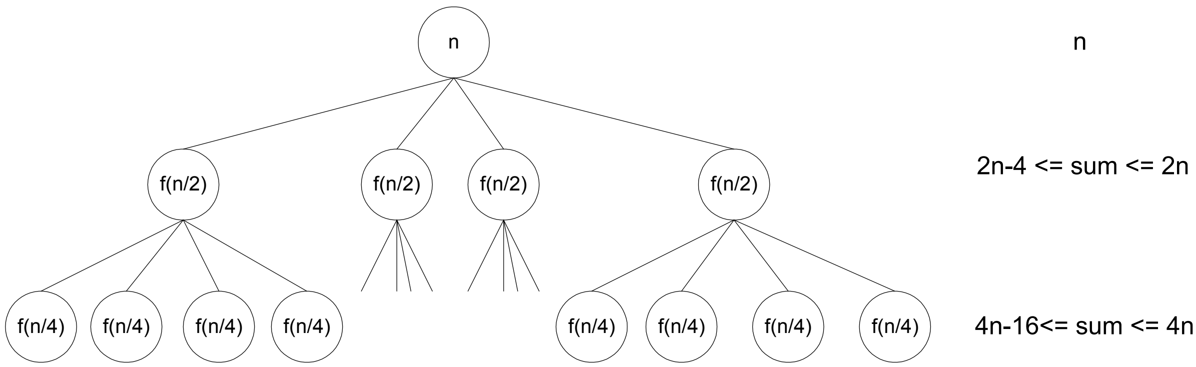
[Skip the substitution method]

4.2-2



The shortest path from the root to a leaf is n→ (1/3)n → $(1/3)^2/n$ → … → 1
∴ $(1/3)^k$ n = 1 when k = $\log_3$ n
∴ the height of the tree is at least $\log_3$ n, which is in Ω(n lg n).

4.2-3



Note: f(n) means $\lfloor n \rfloor$ and the tree ends at the same level.

$\lfloor n/2^i \rfloor = 1 \Rightarrow i \geq \lg n$ so the height of the tree is lg n (ends at the same level)

The solution is at most:

$$
\begin{aligned}
T(n) &\leq \sum_{i=0}^{\lg n} 2^i n = n \sum_{i=0}^{\lg n} 2^i \\
&= n\left(\frac{2^{\lg n+1}-1}{2-1}\right) = n(2^{\lg n}\cdot 2 - 1) \\
&= 2n(n - 1/2) \\
&\leq 2n^2 \in O(n^2)
\end{aligned}
$$

Similarly, the solution is at least:

$$
\begin{aligned}
T(n) &\geq \sum_{i=0}^{\lg n} (2^i n - 2^{2i}) \\
&= n\left(\frac{2^{\lg n+1}-1}{2-1}\right) - \left(\frac{4^{\lg n+1}-1}{4-1}\right) \\
&= n(2^{\lg n}\cdot 2 - 1) - (4^{\lg n}\cdot 2 - 1)/3 \\
&= n(n^{\lg 2}\cdot 2 - 1) - (n^{\lg 4}\cdot 2 - 1)/3 \\
&= 2n^2 - n - \tfrac{2}{3}n^2 - \tfrac{1}{3} \\
&\geq n^2 - n - 1/3 \in \Omega(n^2)
\end{aligned}
$$

Therefore, $T(n) = \Theta(n^2)$.
[Skip the substitution method]

(Optional) 4.3-1

[Your answer should include a proof that shows how the constants can fit into the theorem]

a) $T(n) = \Theta(n^2)$
b) $\Theta(n^2 \lg n)$
c) $\Theta(n^3)$

(Option) 4.3-3

a=1, b=2, f(n)= Θ(1). Then $n^{\log_b a} = \Theta(1) = f(n)$. Thus, $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\lg n)$.