Lecture 13: All-Pairs Shortest Paths

CLRS Section 25.1

Outline of this Lecture

- Introduction of the all-pairs shortest path problem.

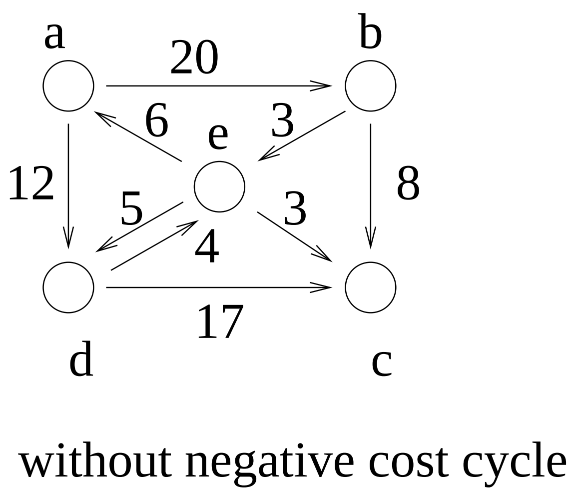
- First solution using Dijkstra's algorithm.
  Assumes no negative weight edges
  $\Theta\left(|V|^3 \log |V|\right)$.
  Needs priority queues

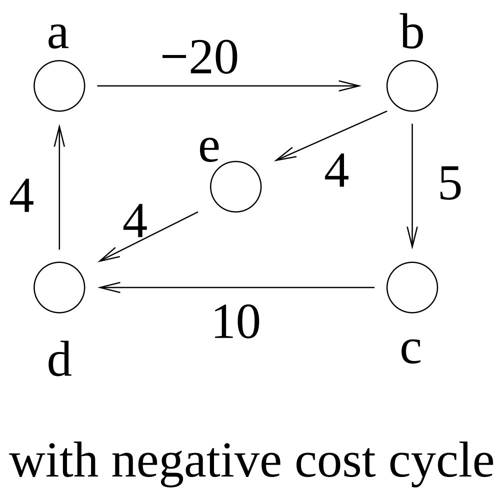
- A (first) dynamic programming solution.
  Only assumes no negative weight cycles.
  First version is $\Theta\left(|V|^4\right)$.
  Repeated squaring reduces to $\Theta\left(|V|^3 \log |V|\right)$.

  No special data structures needed.

The All-Pairs Shortest Paths Problem

Given a weighted digraph $G = (V, E)$ with weight function $w : E \to \mathbf{R}$, (R is the set of real numbers), determine the length of the shortest path (i.e., distance) between all pairs of vertices in G. Here we assume that there are no cycles with zero or negative cost.





Solution 1: Using Dijkstra's Algorithm

If there are no negative cost edges apply Dijkstra's algorithm to each vertex (as the source) of the digraph.

- Recall that D's algorithm runs in $\Theta((n+e)\log n)$. This gives a

  $$\Theta(n(n+e)\log n) = \Theta(n^2 \log n + ne \log n)$$

  time algorithm, where $n = |V|$ and $e = |E|$.

- If the digraph is dense, this is an $\Theta(n^3 \log n)$algorithm.

- With more advanced (complicated) data structures D's algorithm runs in $\Theta(n \log n + e)$time yielding a $\Theta(n^2 \log n + ne)$ final algorithm. For dense graphs this is $\Theta(n^3)$ time.

Solution 2: Dynamic Programming

(1) How do we decompose the all-pairs shortest paths problem into subproblems?

(2) How do we express the optimal solution of a subproblem in terms of optimal solutions to some subsubproblems?

(3) How do we use the recursive relation from (2) to compute the optimal solution in a bottom-up fashion?

(4) How do we construct all the shortest paths?

Solution 2: Input and Output Formats

To simplify the notation, we assume that $V = \{1, 2, \ldots, n\}$.

Assume that the graph is represented by an $n \times n$ matrix with the weights of the edges:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ w(i,j) & \text{if } i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

Output Format: an $n \times n$ matrix $D = [d_{ij}]$ where d_{ij} is the length of the shortest path from vertex i to j.

Step 1: How to Decompose the Original Problem

- Subproblems with smaller sizes should be easier to solve.

- An optimal solution to a subproblem should be expressed in terms of the optimal solutions to subproblems with smaller sizes.

These are guidelines ONLY.

Step 1: Decompose in a Natural Way

- Define $d_{ij}^{(m)}$ to be the length of the shortest path from i to j that contains at most m edges.
  Let $D^{(m)}$ be the $n \times n$ matrix $[d_{ij}^{(m)}]$.

- $d_{ij}^{(n-1)}$ is the true distance from i to j (see next page for a proof this conclusion).

- Subproblems: compute $D^{(m)}$ for $m = 1, \cdots, n-1$.

Question: Which $D^{(m)}$ is easiest to compute?

$d_{ij}^{(n-1)}$ **= True Distance from i to j**

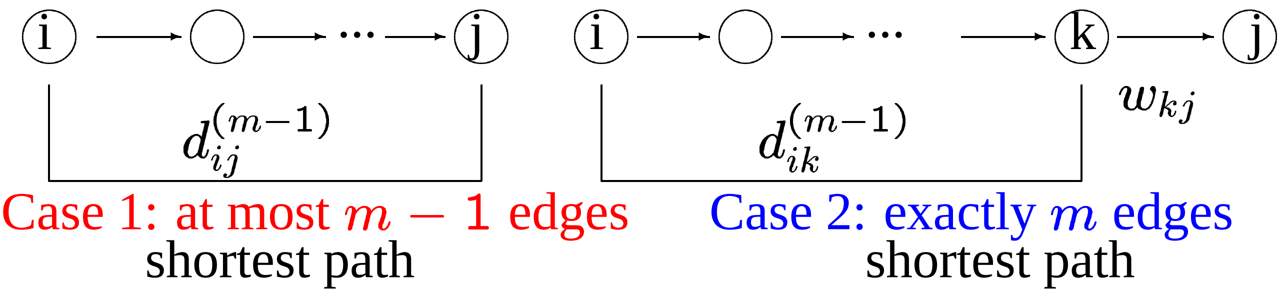
Proof: We prove that any shortest path P from i to j contains at most $n-1$ edges.

First note that since all cycles have positive weight, a shortest path can have no cycles (if there were a cycle, we could remove it and lower the length of the path).

A path without cycles can have length at most $n-1$ (since a longer path must contain some vertex twice, that is, contain a cycle).

A Recursive Formula

Consider a shortest path from i to j of length $d_{ij}^{(m)}$.



Case 1: It has at most $m-1$ edges.
Then $d_{ij}^{(m)} = d_{ij}^{(m-1)} = d_{ij}^{(m-1)} + w_{jj}$.
Case 2: It has m edges. Let k be the vertex before j on a shortest path.
Then $d_{ij}^{(m)} = d_{ik}^{(m-1)} + w_{kj}$.

Combining the two cases,

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

Step 3: Bottom-up Computation of $D^{(n-1)}$

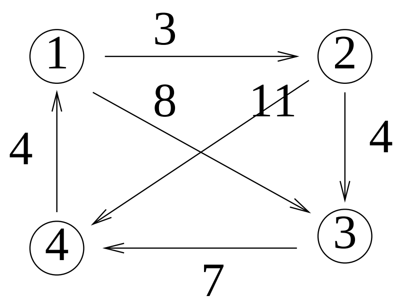
- Bottom: $D^{(1)} = \left[w_{ij}\right]$, the weight matrix.

- Compute $D^{(m)}$ from $D^{(m-1)}$, for $m = 2, ..., n-1$, using

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

Example: Bottom-up Computation of $D^{(n-1)}$

Example

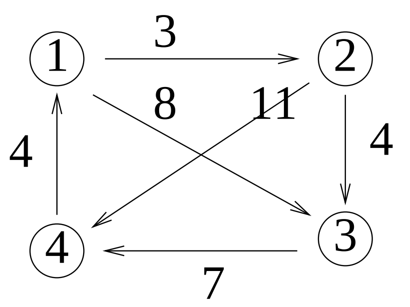


$D^{(1)} = [w_{ij}]$ is just the weight matrix:

$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty \\ \infty & 0 & 4 & 11 \\ \infty & \infty & 0 & 7 \\ 4 & \infty & \infty & 0 \end{bmatrix}$$

Example: Computing $D^{(2)}$ from $D^{(1)}$

$$d_{ij}^{(2)} = \min_{1 \le k \le 4} \left\{ d_{ik}^{(1)} + w_{kj} \right\}.$$

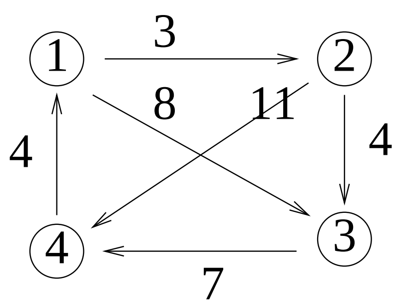


With $D^{(1)}$ given earlier and the recursive formula,

$$D^{(2)} = \begin{bmatrix} 0 & 3 & 7 & 14 \\ 15 & 0 & 4 & 11 \\ 11 & \infty & 0 & 7 \\ 4 & 7 & 12 & 0 \end{bmatrix}$$

Example: Computing $D^{(3)}$ from $D^{(2)}$

$$d_{ij}^{(3)} = \min_{1 \le k \le 4} \left\{ d_{ik}^{(2)} + w_{kj} \right\}$$



With $D^{(2)}$ given earlier and the recursive formula,

$$D^{(3)} = \begin{bmatrix} 0 & 3 & 7 & 14 \\ 15 & 0 & 4 & 11 \\ 11 & 14 & 0 & 7 \\ 4 & 7 & 11 & 0 \end{bmatrix}$$

$D^{(3)}$ gives the distances between any pair of vertices.

The Algorithm for Computing $D^{(n-1)}$

```
for $m = 1$ to $n - 1$
    for $i = 1$ to $n$
        for $j = 1$ to $n$
        {
            $min = \infty$;
            for $k = 1$ to $n$
            {
                $new = d_{ik}^{(m-1)} + w_{kj}$;
                if ($new < min$) $min = new$;
            }
            $d_{ij}^{(m)} = min$;
        }
```

Comments on Solution 2

- Algorithm uses $\Theta(n^3)$ space; how can this be reduced down to $\Theta(n^2)$?

- How can we extract the actual shortest paths from the solution?

- Running time $O(n^4)$, much worse than the solution using Dijkstra's algorithm. Can we improve this?

Repeated Squaring

Observe that we are only interested to find $D^{(n-1)}$, all others D^i, $1 \leq i \leq n-2$ are only auxiliary. Furthermore, since the graph does not have negative cycle, we have $D^{(n-1)} = D^i$, for all $i \geq n$.

In particular, this implies that $D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n-1)}$.

We can calculate $D^{\left(2^{\lceil \log_2 n \rceil}\right)}$ using "repeated squaring" to find

$$D^{(2)},\ D^{(4)}, D^{(8)}, \ldots,\ D^{\left(2^{\lceil \log_2 n \rceil}\right)}$$

We use the recurrence relation:

- Bottom: $D^{(1)} = \left[w_{ij}\right]$, the weight matrix.

- For $s \geq 1$ compute $D^{(2s)}$ using

$$d_{ij}^{(2s)} = \min_{1 \leq k \leq n} \left\{ d_{ik}^{(s)} + d_{kj}^{(s)} \right\}.$$

Given this relation we can calculate $D^{(2^i)}$ from $D^{(2^{i-1})}$ in $O(n^3)$ time. We can therefore calculate all of

$$D^{(2)},\ D^{(4)}, D^{(8)}, \ldots,\ D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n)}$$

in $O(n^3 \log n)$ time, improving our running time.

The Floyd-Warshall Algorithm

Step 1 : Decomposition

Definition: The vertices $v_2, v_3, ..., v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, ..., v_{l-1}, v_l \rangle$.

- Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that *all* intermediate vertices on the path (if any) are in set $\{1, 2, \ldots, k\}$.

  $d_{ij}^{(0)}$ is set to be w_{ij}, i.e., no intermediate vertex. Let $D^{(k)}$ be the $n \times n$ matrix $[d_{ij}^{(k)}]$.

- Claim: $d_{ij}^{(n)}$ is the distance from i to j. So our aim is to compute $D^{(n)}$.

- Subproblems: compute $D^{(k)}$ for $k = 0, 1, \cdots, n$.

Step 2: Structure of shortest paths

Observation 1: A shortest path does not contain the same vertex twice. Proof: A path containing the same vertex twice contains a cycle. Removing cycle gives a shorter path.

Observation 2: For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1, 2, \ldots, k\}$, there are two possibilities:

1. k is not a vertex on the path,
The shortest such path has length $d_{ij}^{(k-1)}$.

2. k is a vertex on the path.
The shortest such path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Step 2: Structure of shortest paths

Consider a shortest path from i to j containing the vertex k. It consists of a subpath from i to k and a subpath from k to j.
Each subpath can only contain intermediate vertices in $\{1, ..., k-1\}$, and must be as short as possible, namely they have lengths $d_{ik}^{(k-1)}$ and $d_{kj}^{(k-1)}$.

Hence the path has length $d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$.

Combining the two cases we get

$$d_{ij}^{(k)} = \min\left\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\}.$$

Step 3: the Bottom-up Computation

- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

  for $k = 1, ..., n$.

The Floyd-Warshall Algorithm: Version 1

Floyd-Warshall(w, n)

```
{ for i = 1 to n do                          initialize
     for j = 1 to n do
     { D^0[i, j] = w[i, j];
       pred[i, j] = nil;
     }

  for k = 1 to n do                          dynamic programming
     for i = 1 to n do
        for j = 1 to n do
           if (d^(k-1)[i, k] + d^(k-1)[k, j] < d^(k)[i, j])
                {d^(k)[i, j] = d^(k-1)[i, k] + d^(k-1)[k, j];
                 pred[i, j] = k;}
           else d^(k)[i, j] = d^(k-1)[i, j];
  return d^(n)[1..n, 1..n];
}
```

Comments on the Floyd-Warshall Algorithm

- The algorithm's running time is clearly $\Theta(n^3)$.

- The predecessor pointer `pred[i, j]` can be used to extract the final path (see later).

- Problem: the algorithm uses $\Theta(n^3)$ space.
  It is possible to reduce this down to $\Theta(n^2)$ space by keeping only one matrix instead of n.
  Algorithm is on next page. Convince yourself that it works.

The Floyd-Warshall Algorithm: Version 2

Floyd-Warshall(w, n)

```
{ for i = 1 to n do                          initialize
    for j = 1 to n do
    { d[i,j] = w[i,j];
      pred[i,j] = nil;
    }

  for k = 1 to n do                          dynamic programming
    for i = 1 to n do
      for j = 1 to n do
        if (d[i,k] + d[k,j] < d[i,j])
              {d[i,j] = d[i,k] + d[k,j];
               pred[i,j] = k;}
  return d[1..n, 1..n];
}
```

Extracting the Shortest Paths

The predecessor pointers `pred[i, j]` can be used to extract the final path. The idea is as follows.

Whenever we discover that the shortest path from i to j passes through an intermediate vertex k, we set $pred[i, j] = k$.

If the shortest path does not pass through any intermediate vertex, then $pred[i, j] = nil$.

To find the shortest path from i to j, we consult $pred[i, j]$. If it is nil, then the shortest path is just the edge (i, j). Otherwise, we recursively compute the shortest path from i to $pred[i, j]$ and the shortest path from $pred[i, j]$ to j.

The Algorithm for Extracting the Shortest Paths

```
Path(i, j)
{
    if (pred[i, j] = nil)    single edge
        output (i, j);
    else            compute the two parts of the path
    {
        Path(i, pred[i, j]);
        Path(pred[i, j], j);
    }
}
```

Example of Extracting the Shortest Paths

Find the shortest path from vertex 2 to vertex 3.

2..3	Path$(2,3)$	$pred[2,3] = 4$	
2..4..3	Path$(2,4)$	$pred[2,4] = 5$	
2..5..4..3	Path$(2,5)$	$pred[2,5] = nil$	$Output$(2,5)
25..4..3	Path$(5,4)$	$pred[5,4] = nil$	$Output$(5,4)
254..3	Path$(4,3)$	$pred[4,3] = 6$	
254..6..3	Path$(4,6)$	$pred[4,6] = nil$	$Output$(4,6)
2546..3	Path$(6,3)$	$pred[6,3] = nil$	$Output$(6,3)
25463			