HMM / SFSA / WFSA

Decoding, Evaluation, and Learning for

Hidden Markov Models & Stochastic/Weighted Finite State Automata

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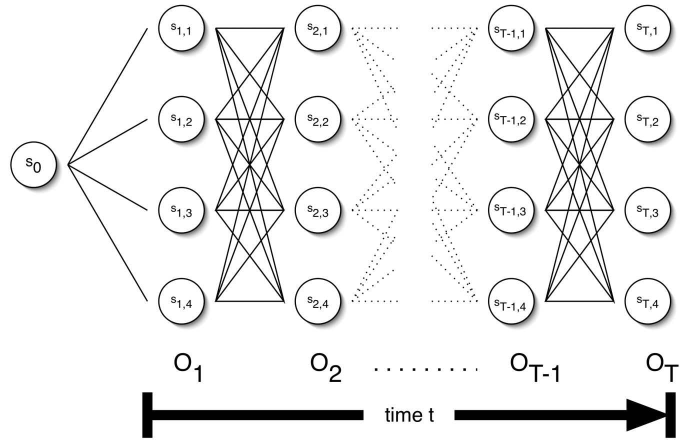
The Trellis



Markov Assumption

- The Markov assumption states that probability of the occurrence of an observed token o_t at time t depends only on the occurrence of o_{t-1} at time t-1
  - Chain rule:

$$P(\mathbf{o}) = P(o_{0..T}) = P(o_0,\ldots,o_{T-1}) = P(o_0)\prod_{t=1}^{T-1} P(o_t \mid o_0,\ldots,o_{t-1})$$

  - Markov assumption:

$$P(\mathbf{o}) = P(o_{0..T}) = P(o_0,\ldots,o_{T-1}) \approx P(o_0)\prod_{t=1}^{T-1} P(o_t \mid o_{t-1})$$

Parameters of an HMM

- States: a set of state nodes $\mathbf{n}=n_0,\ldots,n_{N-1}$
- Transition probabilities: $\mathbf{a}= a_{0,0},a_{0,1},\ldots,a_{N-1,N-1}$ where each $a_{i,j}$ represents the probability of transitioning to n_j, given that we're coming from state n_i
- Emission probabilities: a set $\mathbf{b}$ of functions of the form $b_i(o_t)$ which is the probability of observation o_t being emitted by n_i at time t
- Initial state distribution: π_i is the probability that n_i is a start state

The Three Basic HMM Problems

- Problem 1: **Decoding**. Given the observation sequence **o**=$o_{0..T}$=$o_0,\ldots,o_{T-1}$ and an HMM model $\lambda = (\mathbf{a},\mathbf{b},\boldsymbol{\pi})$, how do we find the state sequence that best explains the observations?
- Problem 2: **Evaluation**. Given the observation sequence **o**=$o_{0..T}$ and an HMM model $\lambda = (\mathbf{a},\mathbf{b},\boldsymbol{\pi})$, how do we compute the probability of **o** given the model?

Problem 1: Decoding

- For Problem 1, we want to find the path with the highest probability.
- We want to find the state sequence **q**=$q_{0..T}$=$q_0\ldots q_{T-1}$, such that $\mathbf{q} = \operatorname*{argmax}_{\mathbf{q}'} P(\mathbf{q}' \mid \mathbf{o}, \lambda)$
- Naïve computation is very expensive. Given T observations and N states, there are N^T possible state sequences.
- Even small HMMs, e.g. T=10 and N=10, contain 10 billion different paths
- Solution: use dynamic programming

The Three Basic HMM Problems

- Problem 3: **Learning**. How do we adjust the model parameters $\lambda = (\mathbf{a},\mathbf{b},\boldsymbol{\pi})$, so as to maximize $P(\mathbf{o} \mid \lambda)$?

Viterbi Algorithm

- Similar to computing the forward probabilities, but instead of summing over transitions from incoming states, compute the maximum
- Forward: $$\alpha_t(j) = \left[\sum_{i=0}^{N-1} \alpha_{t-1}(i) a_{ij}\right] b_j(o_t)$$
- Viterbi Recursion:

$$\delta_t(j) = \left[\max_{0 \le i < N} \delta_{t-1}(i) a_{ij}\right] b_j(o_t)$$

Viterbi Algorithm

- Initialization: $\delta_0(j) = \pi_j b_j(o_0) \quad 0 \le j < N$
- Induction:

$$\delta_t(j) = \left[\max_{0 \le i < N} \delta_{t-1}(i) a_{ij}\right] b_j(o_t)$$

$$\psi_t(j) = \left[\arg\max_{0 \le i < N} \delta_{t-1}(i) a_{ij}\right] \quad 0 < t < T, 0 \le j < N$$

- Termination: $p^* = \max_{0 \le i < N} \delta_{T-1}(i) \quad q^*_{T-1} = \arg\max_{0 \le i < N} \delta_{T-1}(i)$
- Reconstruction: $q^*_t = \psi_{t+1}(q^*_{t+1}) \quad t = T-2, \ldots, 0$

Problem 2: Probability of an Observation Sequence

- What is $P(\mathbf{o} \mid \lambda)$?
- The probability of a observation sequence is the sum of the probabilities of all possible state sequences in the HMM, i.e, the sum over all paths that generate $\mathbf{o}$ through an HMM efficiently.
- The solution to Problem 1 (decoding) gives us the max over all paths that generate $\mathbf{o}$ through an HMM efficiently.

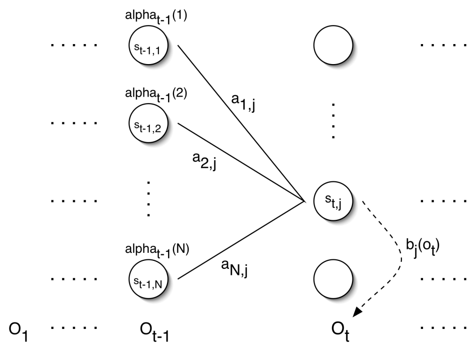
Forward Probabilities

- What is the probability that, given an HMM λ, at time t the state is i and the partial observation $o_0 \ldots o_t$ has been generated?

$$\alpha_t(i) = P(o_{0..t+1}, q_t = n_i \mid \lambda) = P(o_0 \ldots o_t, q_t = n_i \mid \lambda)$$

Forward Probabilities

$$\alpha_t(i) = P(o_0 \ldots o_t, q_t = n_i \mid \lambda)$$



$$\alpha_t(j) = \left[\sum_{i=0}^{N-1} \alpha_{t-1}(i) a_{ij}\right] b_j(o_t)$$

Forward Algorithm

- Initialization: $\alpha_0(j) = \pi_j b_j(o_0) \quad 0 \le j < N$

- Induction:

$$\alpha_t(j) = \left[\sum_{i=0}^{N-1} \alpha_{t-1}(i) a_{ij} \right] b_j(o_t) \quad 0 < t < T, 0 \le j < N$$

- Termination: $P(\mathbf{o} \mid \lambda) = \sum_{i=0}^{N-1} \alpha_{T-1}(i)$

Forward Algorithm Complexity

- In the naïve approach to solving problem 1 it takes on the order of $2T \times N^T$ computations
- The forward algorithm takes on the order of $N^2 T$ computations

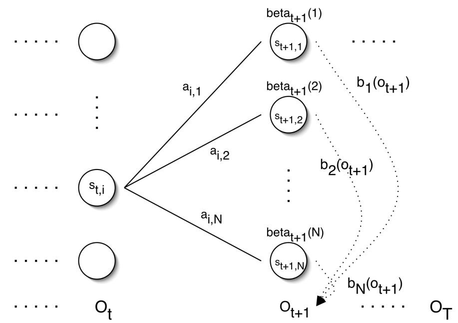
Backward Probabilities

- Or, we can instead compute right-to-left, analogous to the forward probability, but in the opposite direction
- What is the probability that, given an HMM λ and given the state at time t is i, the partial observation $o_{t+1..T}$ is generated?

$$\beta_t(i) = P(o_{t+1..T} \mid q_t = n_i, \lambda) = P(o_{t+1} \ldots o_{T-1} \mid q_t = n_i, \lambda)$$

Backward Probabilities

$$\beta_t(i) = P(o_{t+1} \ldots o_T \mid q_t = n_i, \lambda)$$



$$\beta_t(i) = \left[\sum_{j=0}^{N-1} a_{ij} b_j(o_{t+1}) \beta_{t+1}(j) \right]$$

Backward Algorithm

- Initialization: $\beta_{T-1}(i) = 1, \quad 0 \le i < N$
- Induction:

$$\beta_t(i) = \left[\sum_{j=0}^{N-1} a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)\right] \quad t = T-2...0, 0 \le i < N$$

- Termination:

$$P(\mathbf{o} \mid \lambda) = \sum_{i=0}^{N-1} \pi_i \, \beta_0(i)$$

Problem 3: Learning

- Unfortunately, there is no known way to analytically find a global maximum, i.e., a model λ', such that $\lambda' = \arg\max_{\lambda} P(\mathbf{o} \mid \lambda)$
- But it is possible to find a *local* maximum via the expectation-maximization (EM) approach
- Given an initial model λ, we can always find a model λ', such that $P(\mathbf{o} \mid \lambda') \ge P(\mathbf{o} \mid \lambda)$

Problem 3: Learning

- Up to now we've assumed that we know the underlying model $\lambda = (\mathbf{a}, \mathbf{b}, \boldsymbol{\pi})$
- Often these parameters are estimated on annotated training data, which has two drawbacks:
  - Annotation is difficult and/or expensive
  - Training data is different from the current data
- We want to maximize the parameters with respect to the current data, i.e., we're looking for a **maximum likelihood** model λ^*, such that

$$\lambda^* = \operatorname*{argmax}_{\lambda} P(\mathbf{o} \mid \lambda)$$

Parameter Re-estimation

- Use the **forward-backward algorithm** (a.k.a. the Baum-Welch algorithm), which is a hill-climbing algorithm
- Using an initial parameter instantiation, the forward-backward algorithm iteratively re-estimates the parameters, and improves the probability that the given observations are generated by the new parameters

Parameter Re-estimation

- Three parameters need to be re-estimated:
  - Initial state distribution: π_i
  - Transition probabilities: $a_{i,j}$
  - Emission probabilities: $b_i(o_t)$

Re-estimating Transition Probabilities

- Expected number of transitions:

  what's the probability of being in state n_i at time t and going to state n_j, given the current model and parameters?

$$\xi_t(i,j) = P(q_t = n_i, q_{t+1} = n_j \mid \mathbf{o}, \lambda)$$

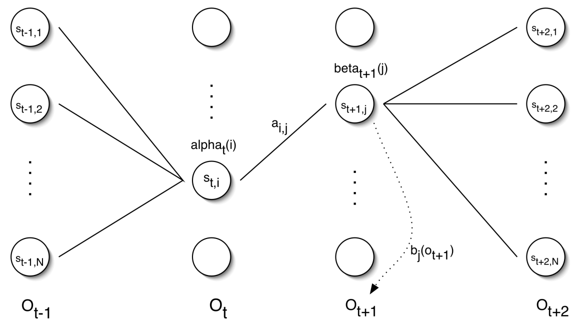
Re-estimating Transition Probabilities

- The key intuition is that we want to compute our expected fractional counts on the number of times each transition is traversed, and then normalize:

$$\hat{a}_{i,j} = \frac{\text{expected number of transitions from state } n_i \text{ to state } n_j}{\text{expected number of transitions from state } n_i}$$

Re-estimating Transition Probabilities

$$\xi_t(i,j) = P(q_t = n_i, q_{t+1} = n_j \mid \mathbf{o}, \lambda)$$



$$\xi_t(i,j) = \frac{P(q_t = n_i, q_{t+1} = n_j, \mathbf{o} \mid \lambda)}{P(\mathbf{o} \mid \lambda)} = \frac{\alpha_t(i)\, a_{i,j}\, b_j(o_{t+1})\, \beta_{t+1}(j)}{P(o_{0..T} \mid \lambda)} = \frac{\alpha_t(i)\, a_{i,j}\, b_j(o_{t+1})\, \beta_{t+1}(j)}{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \alpha_t(i)\, a_{i,j}\, b_j(o_{t+1})\, \beta_{t+1}(j)}$$

Re-estimating Transition Probabilities

- Remember, the intuition behind the re-estimation equation for transition probabilities is

$$\hat{a}_{i,j} = \frac{\text{expected number of transitions from state } \mathrm{n_i} \text{ to state } \mathrm{n_j}}{\text{expected number of transitions from state } \mathrm{n_i}}$$

- Formally:

$$\hat{a}_{i,j} = \frac{\sum_{t=0}^{T-2} \xi_t(i,j)}{\sum_{j'=0}^{N-1}\sum_{t=0}^{T-2} \xi_t(i,j')}$$

Re-estimating Transition Probabilities

- We can rewrite this more neatly as

$$\hat{a}_{i,j} = \frac{\sum_{t=0}^{T-2} \xi_t(i,j)}{\sum_{t=0}^{T-2} \gamma_t(i)}$$

where we define $\gamma_t(i) = \sum_{j=0}^{N-1} \xi_t(i,j)$

to be the probability of being in state n_i at time t, given the complete observation **o**

Review of Probabilities

- Forward probability: $\alpha_t(i)$
  The probability of being in state n_i, given the partial observation $o_0,\ldots,o_t$
- Backward probability: $\beta_t(i)$
  The probability of being in state n_i, given the partial observation $o_{t+1},\ldots,o_{T-1}$
- Transition probability: $\xi_t(i,j)$
  The probability of going from state n_i, to state n_j, given the complete observation $o_0,\ldots,o_{T-1}$
- State probability: $\gamma_t(i)$
  The probability of being in state n_i, given the complete observation $o_0,\ldots,o_{T-1}$

Re-estimating Initial State Probabilities

- Initial state distribution: π_i is the probability that n_i is a start state
- Re-estimation is easy:

$$\hat{\pi}_i = \text{expected number of times in state } \mathrm{n_i} \text{ at time } 0$$

- Formally: $\hat{\pi}_i = \gamma_0(i)$

Re-estimation of Emission Probabilities

- Emission probabilities are re-estimated as

$$\hat{b}_i(k) = \frac{\text{expected number of times in state } n_i \text{ and observe symbol } w_k}{\text{expected number of times in state } n_i}$$

- Formally:

$$\hat{b}_i(k) = \frac{\sum_{t=0}^{T-1} \delta(o_t, w_k)\gamma_t(i)}{\sum_{t=0}^{T-1} \gamma_t(i)}$$

Where $\delta(o_t, w_k) = 1, \text{ if } o_t = w_k, \text{ and 0 otherwise}$

Note that δ here is the Kronecker delta function and is not related to the δ in the discussion of the Viterbi algorithm!!

To iteratively update the model

- Coming from $\lambda = (\mathbf{a}, \mathbf{b}, \boldsymbol{\pi})$ we get to $\lambda' = (\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\boldsymbol{\pi}})$ by the following update rules:

$$\hat{a}_{i,j} = \frac{\sum_{t=0}^{T-2} \xi_t(i,j)}{\sum_{t=0}^{T-2} \gamma_t(i)} \qquad \hat{b}_i(k) = \frac{\sum_{t=0}^{T-1} \delta(o_t, w_k)\gamma_t(i)}{\sum_{t=0}^{T-1} \gamma_t(i)} \qquad \hat{\pi}_i = \gamma_0(i)$$

Expectation-Maximization

- The forward-backward algorithm is an instance of the more general EM algorithm
  - The E Step: Compute the forward and backward probabilities for a given model
  - The M Step: Re-estimate the model parameters

Hidden Markov Models

Bonnie Dorr **Christof Monz**

CMSC 723: Introduction to Computational Linguistics

Lecture 5

October 6, 2004

HMMs and their Usage

- HMMs are very common in Computational Linguistics:
  - Speech recognition (observed: acoustic signal, hidden: words)
  - Handwriting recognition (observed: image, hidden: words)
  - Part-of-speech tagging (observed: words, hidden: part-of-speech tags)
  - Machine translation (observed: foreign words, hidden: words in target language)

Hidden Markov Model (HMM)

- HMMs allow you to estimate probabilities of unobserved events
- Given plain text, which underlying parameters generated the surface
- E.g., in speech recognition, the observed data is the acoustic signal and the words are the hidden parameters

Noisy Channel Model

- In speech recognition you observe an acoustic signal ($A=a_1,\ldots,a_n$) and you want to determine the most likely sequence of words ($W=w_1,\ldots,w_n$): $P(W \mid A)$
- Problem: A and W are too specific for reliable counts on observed data, and are very unlikely to occur in unseen data

Noisy Channel Model

- Assume that the acoustic signal (A) is already segmented wrt word boundaries
- P(W I A) could be computed as

$$P(W \mid A) = \prod_{a_i} \max_{w_i} P(w_i \mid a_i)$$

- Problem: Finding the most likely word corresponding to a acoustic representation depends on the context
- E.g., /'pre-z&ns / could mean "presents" or "presence" depending on the context

Noisy Channel in a Picture

Noisy Channel Model

- Given a candidate sequence W we need to compute P(W) and combine it with P(W I A)
- Applying Bayes' rule:

$$\arg\max_{W} P(W \mid A) = \arg\max_{W} \frac{P(A \mid W) P(W)}{P(A)}$$

- The denominator P(A) can be dropped, because it is constant for all W

Decoding

The decoder combines evidence from

- The likelihood: P(A I W)
  This can be approximated as:

$$P(A \mid W) \approx \prod_{i=1}^{n} P(a_i \mid w_i)$$

- The prior: P(W)
  This can be approximated as:

$$P(W) \approx P(w_1) \prod_{i=2}^{n} P(w_i \mid w_{i-1})$$

Search Space

- Given a word-segmented acoustic sequence list all candidates

'bot	ik-'spen-siv	'pre-z&ns
boat $P('bot \mid bald)$	excessive	presidents
bald $P(inactive \mid bald)$	expensive	presence
bold	expressive	presents
bought	inactive	press

- Compute the most likely path