Linear Programming Duality

P&S Chapter 3

Last Revised – Nov 1, 2004
In this section we lean about duality, which is another way to approach linear programming. In particular, we will see:

- How to define the dual of a normal (primal) linear program. If the primal is a minimization problem, the dual is a maximization problem.

- That the optimum solution to the dual has cost equal to that of the optimum solution to the primal.

- Complementary Slackness: A combinatorial statement of the relationship between the primal and the dual.

- A primal-dual interpretation of the shortest path problem.
Convert an LP in general form such as one on the right into a LP in standard form using method seen earlier. That is

\[
\begin{align*}
\text{min } c'x \\
a_i'x &= b_i \quad i \in M \\
a_i'x &\geq b_i \quad i \in \bar{M} \\
x_j &\geq 0 \quad j \in N \\
x_j &\geq 0 \quad j \in \bar{N}
\end{align*}
\]

where

\[
\begin{align*}
\min \hat{c}'\hat{x} \\
\hat{A}\hat{x} &= b \\
\hat{x} &\geq 0
\end{align*}
\]

where

\[
\begin{align*}
\hat{A} &= \left[ A_j, j \in N \right| (A_j, -A_j), j \in \bar{N} \right| \begin{bmatrix} 0, i \in M \\ -I, i \in \bar{M} \end{bmatrix} \\
\hat{x} &= \text{col}(x_j, j \in N|(x^+_j, x^-_j), j \in \bar{N}|x^s_i, i \in \bar{M}) \\
\hat{c} &= \text{col}(c_j, j \in N|(c_j, -c_j), j \in \bar{N}|0)
\end{align*}
\]
\[
\begin{align*}
\min \ c' \hat{x} & \\
\hat{A} \hat{x} = b & \\
\hat{x} \geq 0 & \\
\end{align*}
\]

\[
\hat{A} = \begin{bmatrix} A_j, j \in N | (A_j, -A_j), j \in \bar{N} \\ 0, i \in M \\ -I, i \in \bar{M} \end{bmatrix}
\]

\[
\hat{x} = \text{col}(x_j, j \in N | (x_j^+, x_j^-), j \in \bar{N}|x_i^+, i \in \bar{M})
\]

\[
\hat{c} = \text{col}(c_j, j \in N | (c_j, -c_j), j \in \bar{N}|0)
\]

Recall that BFS \( \hat{x}_0 \) is optimal iff corresponding \( \bar{c} \geq 0 \). If this occurs, there is a corresponding basis \( \hat{B} \) s.t.

\[
\hat{c}' - (\hat{c}'_B \hat{B}^{-1}) \hat{A} \geq 0
\]

so \( \pi' = \hat{c}'_B \hat{B}^{-1} \) is a feasible solution to

\[
\pi' \hat{A} \leq \hat{c}'
\]

where \( \pi \in R^m \) and \( m = |M| + |\bar{M}| \) is # of rows in original LP.

Note that there are three different sets of inequalities, one each for \( j \in N, j \in \bar{N}, \) and \( i \in \bar{M} \).
\[
\begin{align*}
\min \tilde{c}' \hat{x} & \quad \hat{A} = \begin{bmatrix} A_j, j \in N | (A_j, -A_j), j \in \tilde{N} & 0, i \in M \\ -I, i \in \tilde{M} \end{bmatrix} \\
\hat{A} \hat{x} = b & \quad \hat{x} = \text{col}(x_j, j \in N | (x_j^+, x_j^-), j \in \tilde{N} | x_i^s, i \in \tilde{M}) \\
\hat{x} \geq 0 & \quad \hat{c} = \text{col}(c_j, j \in N | (c_j, -c_j), j \in \tilde{N} | 0)
\end{align*}
\]

which we saw leads to \( \pi' \hat{A} \leq \hat{c}' \)

1. If \( j \in N \) then \( \pi' A_j \leq c_j \)

2. If \( j \in \tilde{N} \) then \( \pi' A_j \leq c_j \) and \( -\pi' A_j \leq -c_j \) so \( \pi' A_j = c_j \).

3. If \( i \in \tilde{M} \) then \( -\pi' \leq 0 \) so \( \pi' \geq 0 \).

Using these equations, given a primal LP in general form, we can define a new LP in dual form.

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min c'x )</td>
<td>( \max \pi' b )</td>
</tr>
<tr>
<td>( a_i'x = b_i )</td>
<td>( \pi_i \geq 0 )</td>
</tr>
<tr>
<td>( a_i'x \geq b_i )</td>
<td>( \pi_i \geq 0 )</td>
</tr>
<tr>
<td>( x_j \geq 0 )</td>
<td>( \pi' A_j \leq c_j )</td>
</tr>
<tr>
<td>( x_j \geq 0 )</td>
<td>( \pi' A_j = c_j )</td>
</tr>
</tbody>
</table>
Theorem: If an LP has an optimal solution, so does its dual and, at optimality, their costs are equal.

Proof: Let $x, \pi$ be feasible solutions to the primal and dual. Then

$$c'x \geq \pi'Ax \geq \pi'b \quad (1)$$

Since primal has optimal solution, dual can not have unbounded feasible solutions. We saw before that, if $\tilde{x}_0$ is optimal in primal then $\pi' = \tilde{c}'_B\tilde{B}^{-1}$ is, by construction, feasible in dual. This means dual is bounded and has some feasible solution so, by simplex algorithm, dual has an optimal (bounded) solution (BFS).

The cost of this particular $\pi'$ is

$$\pi'b = \tilde{c}'_B\tilde{B}^{-1}b = \tilde{c}'_B\tilde{x}_0$$

Therefore, by (1), $\pi'$ is optimal in dual.
Primal | Dual
--- | ---
\(\min c'x\) | \(\max \pi'b\)
\(a_i'x = b_i\) \(i \in M\) | \(\pi_i \geq 0\)
\(a_i'x \geq b_i\) \(i \in \bar{M}\) | \(\pi_i \geq 0\)
\(x_j \geq 0\) \(j \in N\) | \(\pi'A_j \leq c_j\)
\(x_j \geq 0\) \(j \in \bar{N}\) | \(\pi'A_j = c_j\)

**Theorem:** The dual of the dual is the primal.

**Proof:** Write dual as

\[
\min \pi'(-b)
\]
\[
(-A'_j)\pi \geq -c_j \quad j \in N
\]
\[
(-A'_j)\pi = -c_j \quad j \in \bar{N}
\]
\[
\pi_i \geq 0 \quad i \in \bar{M}
\]
\[
\pi_i \geq 0 \quad i \in M
\]

and consider it as primal. Then

\[
\max x'(-c)
\]
\[
x_j \geq 0 \quad j \in N
\]
\[
x_j \geq 0 \quad j \in \bar{N}
\]
\[
-a'_i x \leq -b \quad i \in \bar{M}
\]
\[
-a'_i x = -b \quad i \in M
\]
**Theorem:** Given a primal-dual pair, exactly one of the three situations occurring below occurs. The X’s denote situations which can not occur.

<table>
<thead>
<tr>
<th>Dual Primal</th>
<th>Finite optimum</th>
<th>Unbounded</th>
<th>Infeasible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite optimum</td>
<td>1</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Unbounded</td>
<td>X</td>
<td>X</td>
<td>3</td>
</tr>
<tr>
<td>Infeasible</td>
<td>X</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

**Proof:** We already saw that (1) is possible. We also saw that if primal has finite optimum, then the dual must have a finite optimum so the X’s in the first row are also correct. Since the dual of the dual is the primal we find, by symmetry, that the X’s in the first column are correct.

We also saw that any feasible solution to the problem upper bounds the cost of all feasible solutions to the dual so it is impossible for both of them to simultaneously have unbounded solutions. We will see examples of (2) and (3) on the next slide.
<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min x_1 )</td>
<td>( \max \pi_1 + \pi_2 )</td>
</tr>
<tr>
<td>( x_1 + x_2 \geq 1 )</td>
<td>( \pi_1 - \pi_2 = 1 )</td>
</tr>
<tr>
<td>( -x_1 - x_2 \geq 1 )</td>
<td>( \pi_1 - \pi_2 = 0 )</td>
</tr>
<tr>
<td>( x_1 \geq 0 )</td>
<td>( \pi_1 \geq 0 )</td>
</tr>
<tr>
<td>( x_2 \geq 0 )</td>
<td>( \pi_2 \geq 0 )</td>
</tr>
</tbody>
</table>

Note that both the Primal and Dual are infeasible, giving case 2. Now modify the primal so that \( x_1, x_2 \geq 0 \). Then

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
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<tbody>
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<td>( \min x_1 )</td>
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</tr>
<tr>
<td>( x_1 \geq 0 )</td>
<td>( \pi_1 \geq 0 )</td>
</tr>
<tr>
<td>( x_2 \geq 0 )</td>
<td>( \pi_2 \geq 0 )</td>
</tr>
</tbody>
</table>

and the primal is infeasible while the dual is unbounded (case 3). Primal unbounded and dual infeasible follows by flipping the primal and the dual.
Recall the **Diet Problem:**

\[
\begin{align*}
\min & \quad c'x \\
Ax & \geq r \\
x & \geq 0
\end{align*}
\]

\[a_{i,j} = \text{amount of } i\text{th nutrient in a unit of the } j\text{th food} \]
\[i = 1, \ldots, m, \quad j = 1, \ldots, n\]

\[r_i = \text{yearly requirement of } i\text{th nutrient} \]
\[i = 1, \ldots, m\]

\[x_j = \text{yearly consumption of the } j\text{th food (in units)} \]
\[j = 1, \ldots, n\]

\[c_j = \text{cost per unit of the } j\text{th food} \]
\[j = 1, \ldots, n\]

The Dual is

\[
\begin{align*}
\max & \quad \pi'r \\
\pi'A & \leq c' \\
\pi' & \geq 0
\end{align*}
\]

This can be interpreted as saying that a pill-manufacturer wants to sell \(m\) nutrients; pill \(i\) containing one unit of nutrient \(i\) at price \(\pi_i\). Manufacturer wants to maximize revenue. Constraint is that, for each food \(j\), it must be cheaper for customer to satisfy nutritional requirements via pills rather than eating food.
Complementary Slackness Conditions

**Theorem:** A pair \( x, \pi \) respectively feasible in a primal-dual pair are optimal iff

\[
\begin{align*}
    u_i &= \pi_i(a_i'x - b_i) = 0 \quad \text{for all } i \quad \text{(2)} \\
    v_j &= (c_j - \pi'A_j)x_j = 0 \quad \text{for all } j \quad \text{(3)}
\end{align*}
\]

**Proof:** By duality definitions

\( \forall i, u_i \geq 0 \) and \( \forall j, v_j \geq 0 \). Now set

\[
    u = \sum_i u_i \geq 0 \quad \text{and} \quad v = \sum_j v_j \geq 0
\]

Then \( u = 0 \) iff (2) and \( v = 0 \) iff (3).

Now note that

\[
u + v = c'x - \pi'b.
\]

So, (2) and (3) are true iff \( c'x = \pi'b \) which is true iff both \( x \) and \( \pi \) are optimal.
Example

\[ \begin{align*}
& \text{Primal} \\
& \min x_1 + x_2 + x_3 + x_4 + x_5 \\
& 3x_1 + 2x_2 + x_3 = 1 \\
& 5x_1 + x_2 + x_3 + x_4 = 3 \\
& 2x_1 + 5x_2 + x_3 + x_5 = 4 \\
& x_i \geq 0
\end{align*} \]

\[ \begin{align*}
& \text{Dual} \\
& \max \pi_1 + 3\pi_2 + 4\pi_3 \\
& 3\pi_1 + 5\pi_2 + 2\pi_3 \leq 1 \\
& 2\pi_1 + \pi_2 + 5\pi_3 \leq 1 \\
& \pi_1 + \pi_2 + \pi_3 \leq 1 \\
& \pi_2 \leq 1 \\
& \pi_3 \leq 1 \\
& \pi_i \geq 0 \text{ for all } i
\end{align*} \]

Because primal is in standard form we have

\[ \forall i, u_i = \pi_i(a_i^T x - b_i) = 0. \]

We now need to satisfy

\[ \forall j, v_j = (c_j - \pi^T A_j)x_j = 0. \]
\[ \begin{aligned}
\text{min} & \quad x_1 + x_2 + x_3 + x_4 + x_5 \\
& \quad 3x_1 + 2x_2 + x_3 = 1 \\
& \quad 5x_1 + x_2 + x_3 + x_4 = 3 \\
& \quad 2x_1 + 5x_2 + x_3 + x_5 = 4 \\
& \quad x_i \geq 0
\end{aligned} \]

Primal

\[ \begin{aligned}
\text{max} & \quad \pi_1 + 3\pi_2 + 4\pi_3 \\
& \quad 3\pi_1 + 5\pi_2 + 2\pi_3 \leq 1 \\
& \quad 2\pi_1 + \pi_2 + 5\pi_3 \leq 1 \\
& \quad \pi_1 + \pi_2 + \pi_3 \leq 1 \\
& \quad \pi_2 \leq 1 \\
& \quad \pi_3 \leq 1 \\
& \quad \pi_i \geq 0 \quad \text{for all } i
\end{aligned} \]

Dual

We now need to satisfy
\[ \forall j, v_j = (c_j - \pi'A_j)x_j = 0. \]
Optimal primal is \((0, 1/2, 0, 5/2, 3/2)\) (with cost 9/2) so need equality in 2nd, 4th and 5th equations, i.e.,

\[ \begin{aligned}
c_2 - \pi'A_2 &= 0 \quad \text{or} \quad 2\pi_1 + \pi_2 + 5\pi_3 = 1 \\
c_4 - \pi'A_4 &= 0 \quad \text{or} \quad \pi_2 = 1 \\
c_5 - \pi'A_5 &= 0 \quad \text{or} \quad \pi_3 = 1
\end{aligned} \]

which has solution

\[ (\pi_1, \pi_2, \pi_3) = (-5/2, 1, 1). \]

Note that this has same cost 9/2 as primal and is therefore optimal.
The Shortest Path problem & its dual

Let $G = (V, E)$ be a directed graph s.t. every edge $e_j \in E$ has cost $c_j \geq 0$. The shortest path problem is to find a directed path from source $s$ to sink $t$ with minimal cost. We will now see how to write this as an LP and then derive its dual.

The feasible set of this problem is

$$F = \{ \text{sequences } P = (e_{j_1}, \ldots, e_{j_k}) : \text{this sequence is a directed path from } s \text{ to } t \text{ in } G \}$$

with cost $c(P) = \sum_{i=1}^{k} c_{j_i}$.

Now define the node-incidence matrix $A = [a_{i,j}]$,

$$a_{i,j} = \begin{cases} 
+1 & \text{if arc } e_j \text{ leaves node } i \\
-1 & \text{if arc } e_j \text{ enters node } i \\
0 & \text{otherwise}
\end{cases} \quad (i = 1, \ldots, |V| \text{ and } j = 1, \ldots, |E|)$$
\[ a_{ij} = \begin{cases} 
+1 & \text{if arc } e_j \text{ leaves node } i \\
-1 & \text{if arc } e_j \text{ enters node } i \\
0 & \text{otherwise} 
\end{cases} \quad \left( i = 1, \ldots, |V| \text{ and } j = 1, \ldots, |E| \right) \]

\[ A = \begin{bmatrix} 
+1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 \\
-1 & 0 & +1 & +1 & 0 \\
0 & -1 & -1 & 0 & +1 
\end{bmatrix} \]
To create a LP introduce $f_j$ to denote flow through arc $e_j$. The intuition is that we would like to send one unit of flow from $s$ to $t$. The cost of one unit of flow through $e_j$ will be $c_j$ so a shortest $s - t$ path “should” be one that minimizes cost.

We need flow conservation at every non-$s,t$ node $v_i$. This corresponds to $a'_i f = 0$. On the other hand, one net unit of flow leaves $s$ and one net unit enters $t$ so the problem that we want to solve is

\[
\min c' f
\]

\[
Af = \begin{bmatrix}
+1 \\
-1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

where the $+1$ corresponds to row $s$ and the $-1$ corresponds to row $t$.

Note that it is possible that the $f_j$ “could” take on non-integer values but it is easy to see that there is an optimal solution in which each $f_j = 1$ (in shortest path) or $f_j = 0$ (not in shortest path). We will prove this more formally later when discussing unimodular matrices.
\[ \min c'f \]
\[ Af = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]
\[ f \geq 0 \]

where the \(+1\) corresponds to row \(s\) and the \(-1\) corresponds to row \(t\).

Note that the \(|V|\) equations are redundant and we can therefore leave out any one equation. It is most convenient to leave out the row \(t\) equation since this will leave a nonnegative cost column in our simplex tableau.
A basis corresponds to a set of arcs containing a $s - t$ path. Degenerate elements in basis are non-path arcs.
Pivoting lets us remove $f_4$ from the basis and add $f_2$. Since $\bar{c} \geq 0$ we see that solution $(0, 1, 0, 0, 1)$ is feasible optimal so path \{f_2, f_5\} is optimal.
Primal

$$\min c'f$$

$$Af = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$f \geq 0$$

Dual

$$\max \pi_s - \pi_t$$

$$\pi'A \leq c'$$

$$\pi \geq 0$$

Since the column corresponding to $e = (i, j)$ has a 1 in row $i$, a $-1$ in row $j$ and a 0 elsewhere, the dual inequalities can be written as

$$\pi_i - \pi_j \leq c_{ij} \quad \text{for each } (i, j) \in E \quad (4)$$

The complementary slackness conditions then say:

Path $f$ and assignment $\pi$ are jointly optimal iff

(i) each arc $e = (i, j)$ in shortest path, i.e.,

$$f_e > 0$$

corresponds to $\pi_i - \pi_j = c_{ij}$ and

(ii) $\pi_i - \pi_j < c_{ij}$ corresponds to $f_{(i,j)} = 0$, i.e.,

$e = (i, j)$ not in shortest path.
Path $f$ and assignment $\pi$ are jointly optimal iff
(i) each arc $e = (i, j)$ in shortest path, i.e.,
\[ f_e > 0 \] corresponds to $\pi_i - \pi_j = c_{ij}$ and
(ii) $\pi_i - \pi_j < c_{ij}$ corresponds to $f(i, j) = 0$, i.e.,
e $= (i, j)$ not in shortest path.

Let the shortest $s - t$ path found by simplex be
\[ e_k e_{k-1} \cdots e_1, \]
where $e_i = u_i, v_i, u_k = s, v_1 = t$ and $v_i = u_{i-1}$. If $v$ is some node on the path then, by definition,
$\pi_v - \pi_t$ is the shortest distance from $v$ to $t$
so $\pi_s - \pi_t$, the objective maximized by the dual, is exactly the shortest distance from $s$ to $t$. 
The figure shows an optimal $\pi$ corresponding to shortest path

$$(s, b), (b, t)$$

Note that when we calculate $\pi$ we don’t really have a value for $\pi_t$, but it can be calculated from the fact that $f_{b, t} = 1 > 0$ and the complementary slackness conditions which force

$$\pi_t = \pi_b - c_{b, t} = 1 - 1 = 0$$
<table>
<thead>
<tr>
<th>Initial Tableau</th>
<th>Final Tableau</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_j$</td>
<td>$c_j - \pi_j$</td>
</tr>
<tr>
<td>$I$</td>
<td>$B^{-1}$</td>
</tr>
</tbody>
</table>

Assume, WLOG, that simplex starts with Identity matrix in left side, e.g., slack or artificial variables.

At end of algorithm, we have essentially multiplied matrix by $B^{-1}$ where $B$ is the set of columns in original matrix corresponding to final optimal BFS.

At optimality, cost row is

$$0 \leq \bar{c} = c - \left( c'B B^{-1} \right) A = c - \pi' A$$

where we have already seen that $\pi' = c'B B^{-1}$ is an optimal solution to the dual. So,

$$\bar{c}_j = c_j - \pi_j \quad j = 1, \ldots, m$$

and

$$\pi_j = c_j - \bar{c}_j \quad j = 1, \ldots, m$$

and we can read off optimal dual solution from tableau.
Example

In the section on simplex we saw that the tableau on page 13 is equivalent to the two-phase tableau

\[
\begin{array}{cccccccc}
\text{\(-z\)} & x_1^a & x_2^a & x_3^a & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-8 & -10 & -3 & -1 & -1 \\
1 & 3 & 2 & 1 & 0 & 0 \\
3 & 5 & 1 & 1 & 1 & 0 \\
4 & 2 & 5 & 1 & 0 & 1 \\
\end{array}
\]

This has \(c_j = 0\) in real cost row. We also saw that at optimality this transforms to

\[
\begin{array}{cccccccc}
\text{\(-z\)} & x_1^a & x_2^a & x_3^a & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
-9/2 & 5/2 & -1 & -1 & 3/2 & 0 & 3/2 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 3/2 & 1 & 1/2 & 0 & 0 \\
5/2 & -1/2 & 1 & 0 & 7/2 & 0 & 1/2 & 1 & 0 \\
3/2 & -5/2 & 0 & 1 & -11/2 & 0 & -3/2 & 0 & 1 \\
\end{array}
\]

Thus, an optimal dual solution would be

\[\pi = (-5/2, 1, 1)\]

which is exactly what we derived on page 13.