New Results on Binary Comparison Search Trees

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Early version of paper at arxiv.org

Optimal search trees with 2-way comparisons

Marek Chrobak, Mordecai Golin, J. Ian Munro, Neal E. Young

arXiv:1505.00357

Main Result

Constructing Min-Cost Binary Comparison Search Trees

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Wasn’t this completely understood 45 years ago??!!

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Constructing Min-Cost Binary Comparison Search Trees

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Yes and No …

Outline

- History
  - Binary Search Trees
  - Hu-Tucker Trees
  - AKKL Trees
- Optimal Binary Comparison Search Trees with Failures
  - Problem Models
  - List of New Results
- New Results
  - The Main Lemma
  - Structural Properties of OBCSTs
  - Dynamic Programming for OBCSTs
  - Proof of The Main Lemma (Sketch)
- Extensions and Open Problems

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- Input: probability of successful and unsuccessful searches

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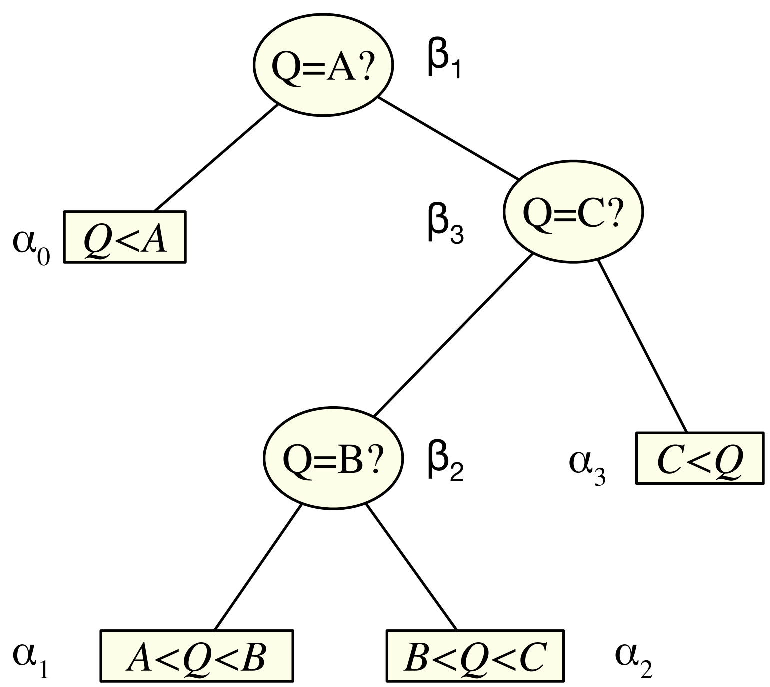

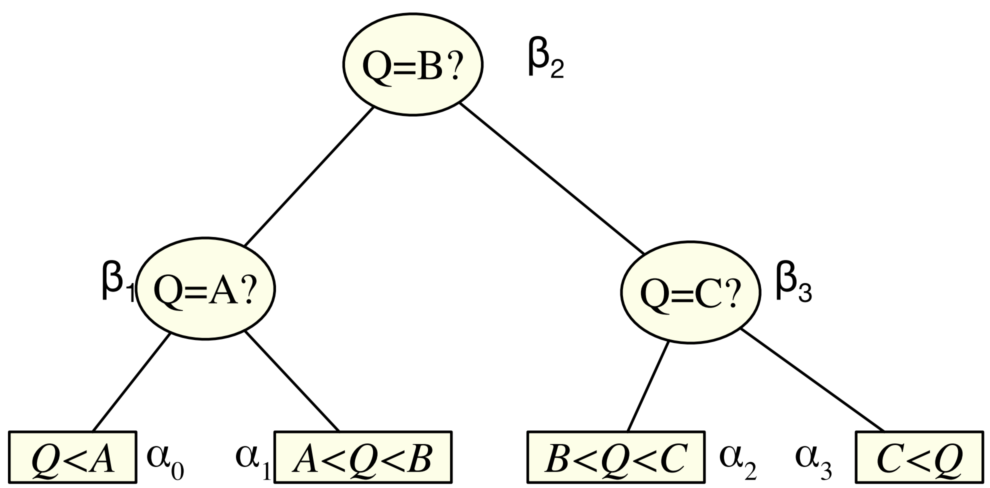

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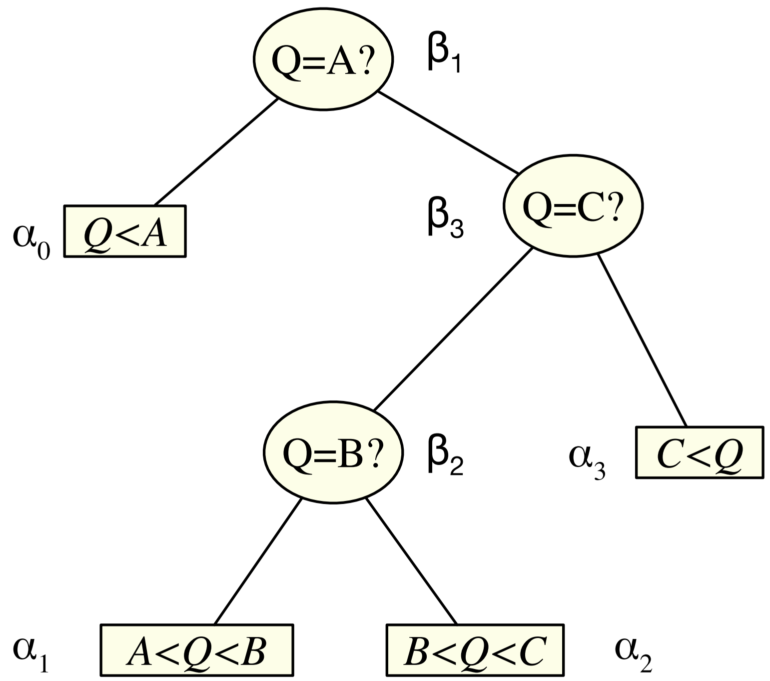

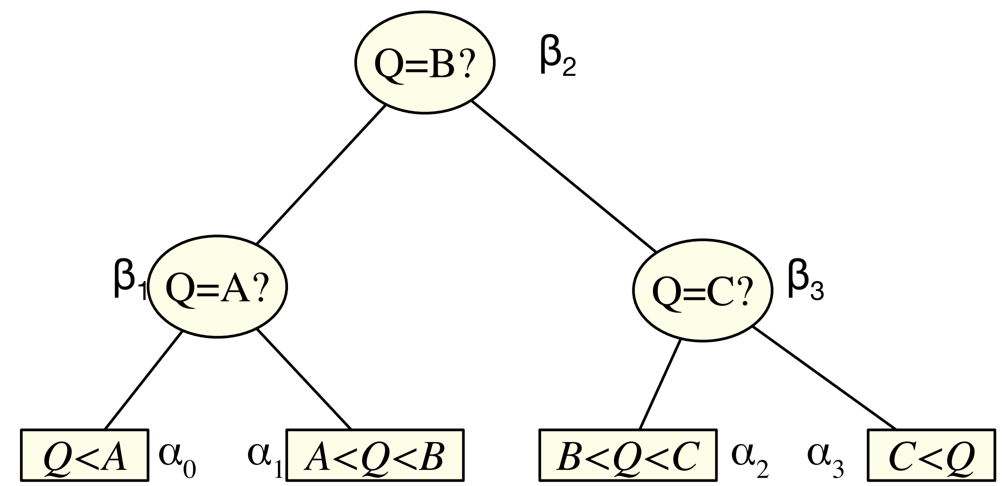

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- Dynamic Programming Algorithm
  - Constructed O(n^2) DP table
    - Knuth reduced O(n^3) running time to O(n^2)
    - Technique later generalized as Quadrangle Inequality method by F. Yao

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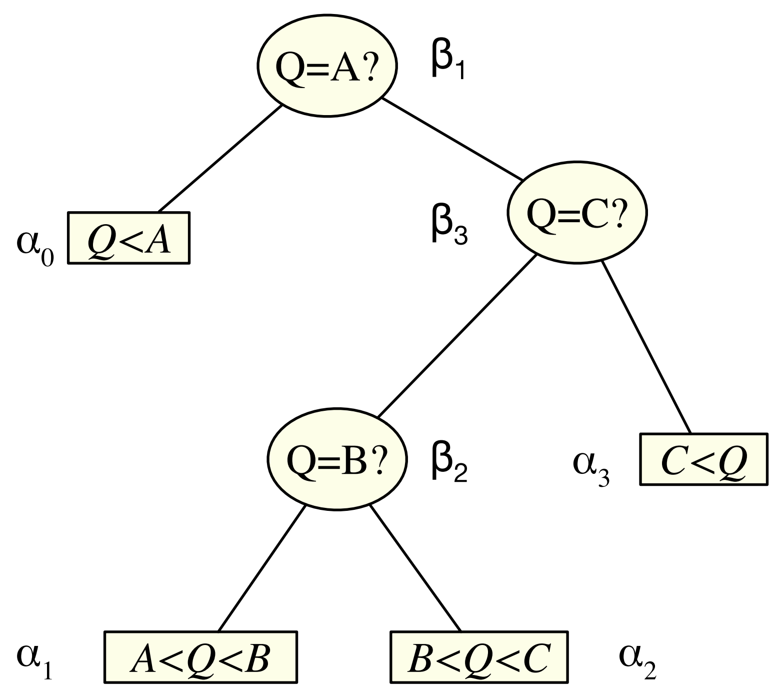

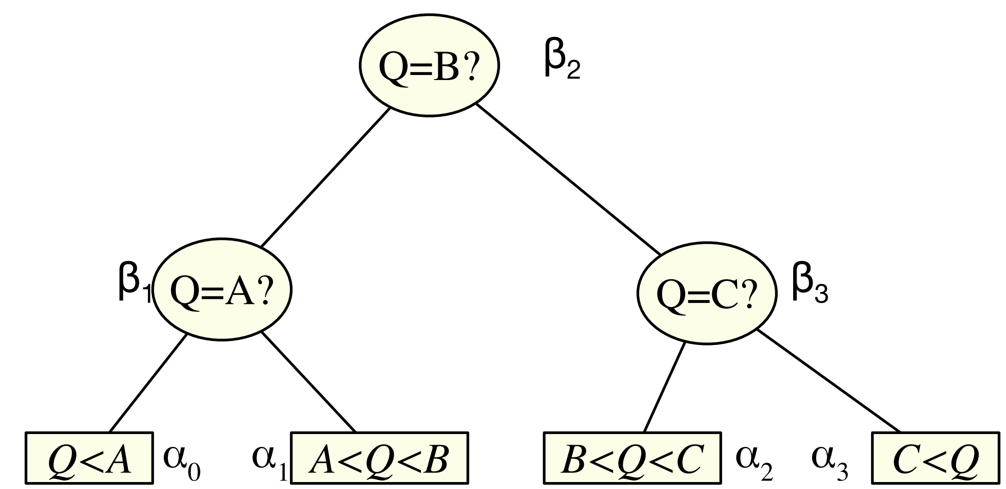

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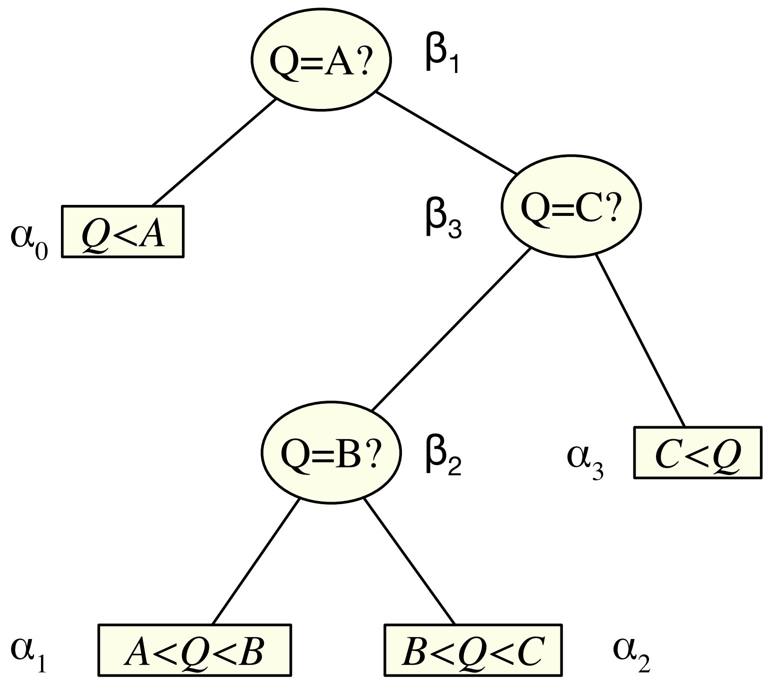

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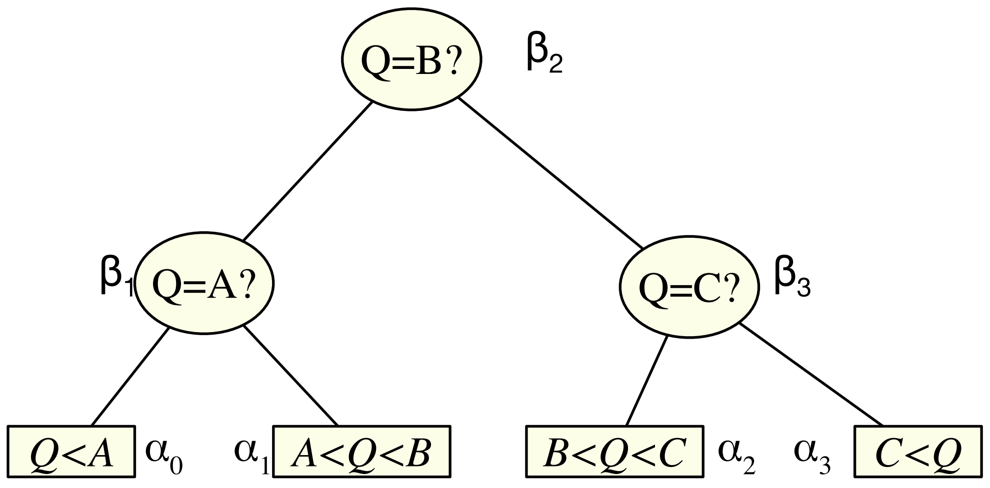

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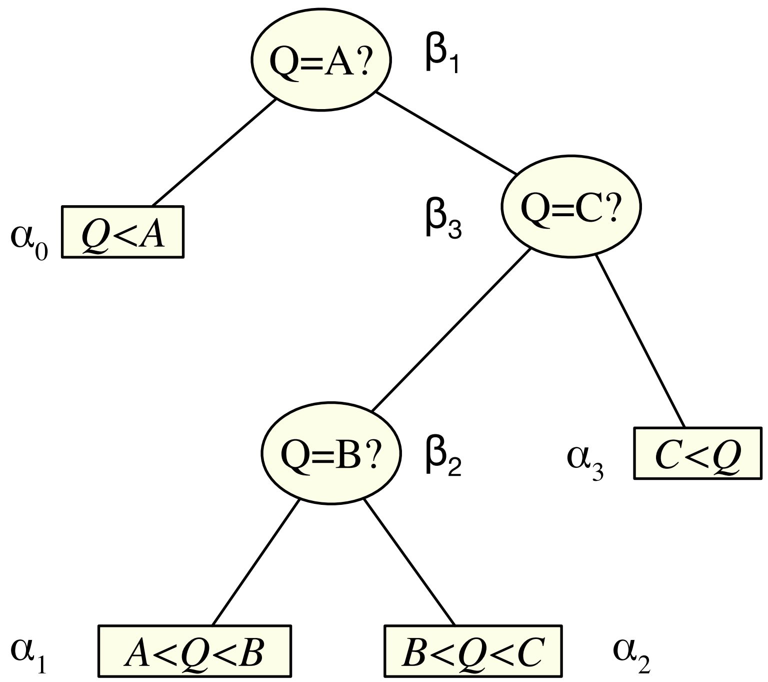

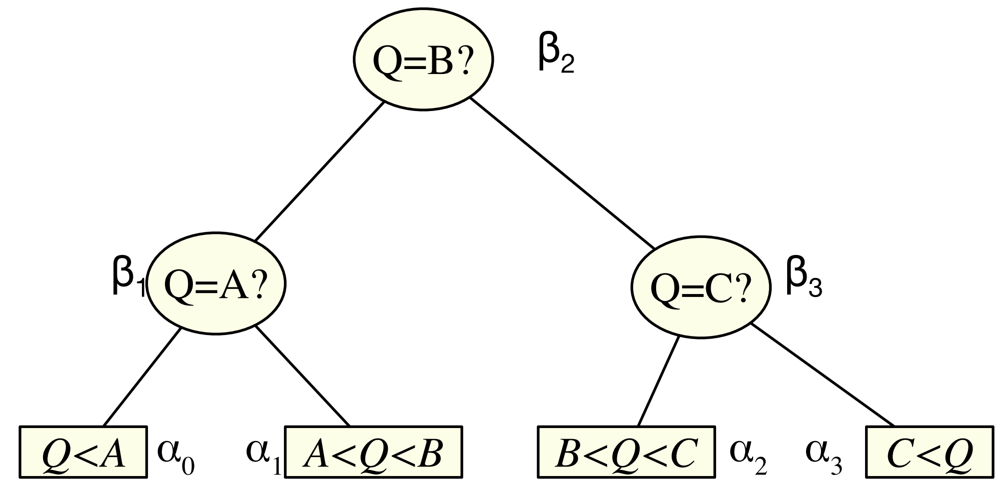

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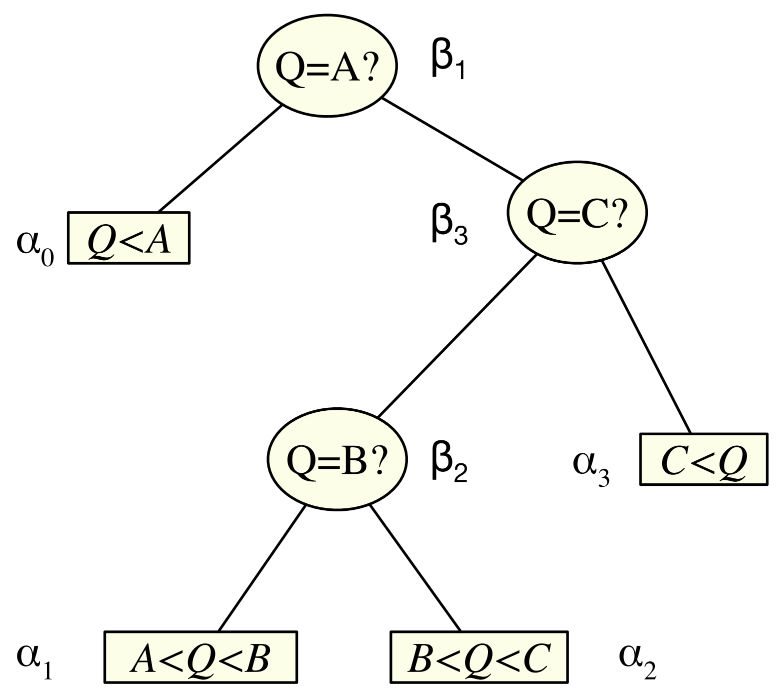

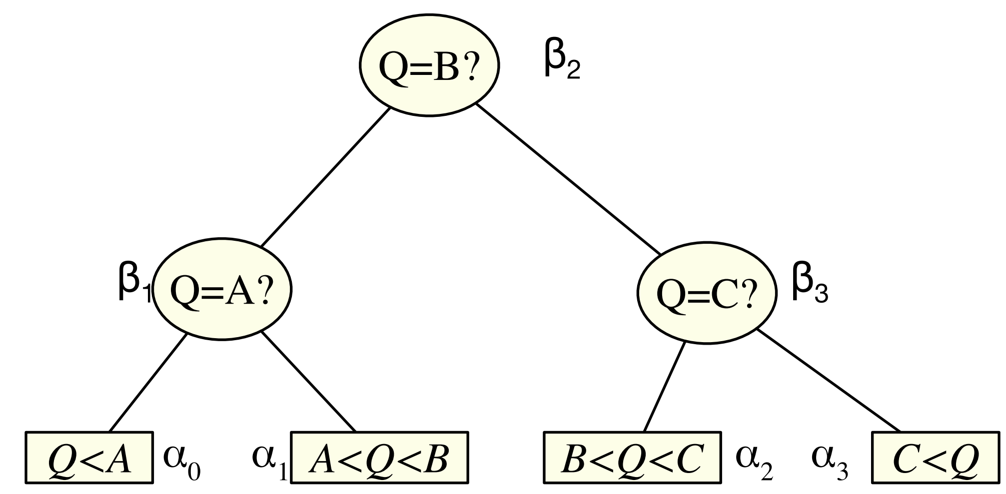

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$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.7, 0.1, 0.1, 0.1)$

$(\beta_1, \beta_2, \beta_3) = (.3, .3, .3)$

Cost = 1.05

Cost = 0.80

Hu-Tucker Binary Comparison Search Trees

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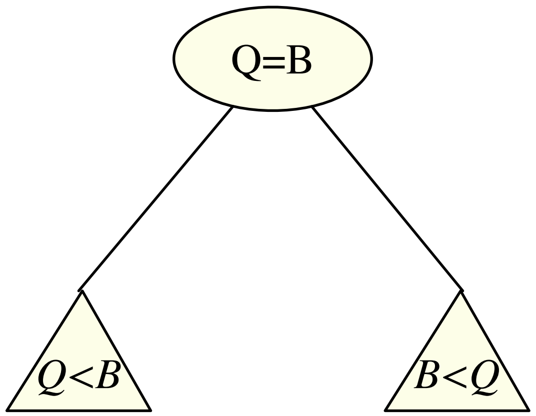

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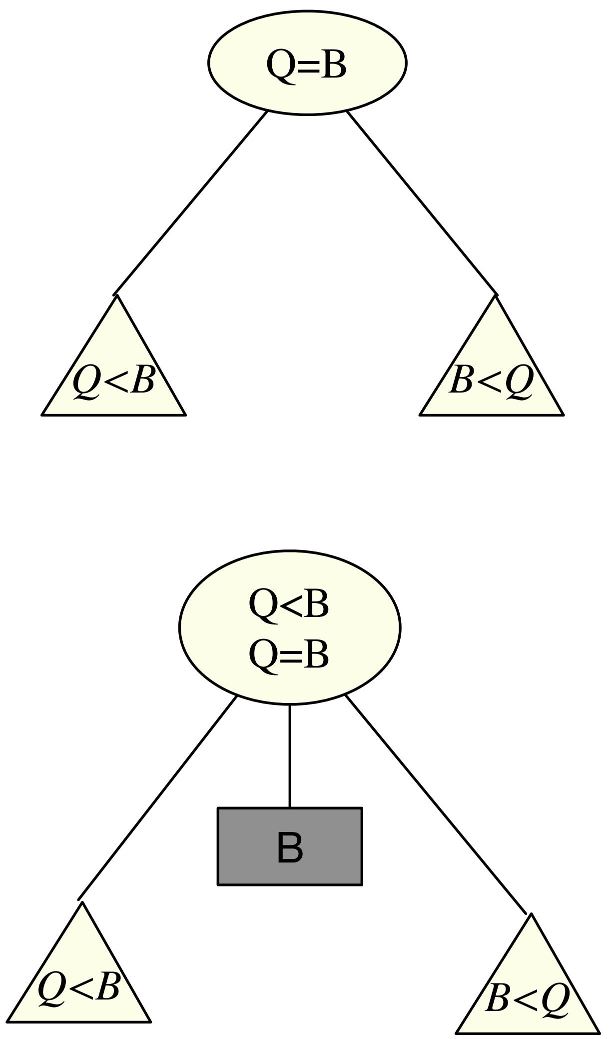

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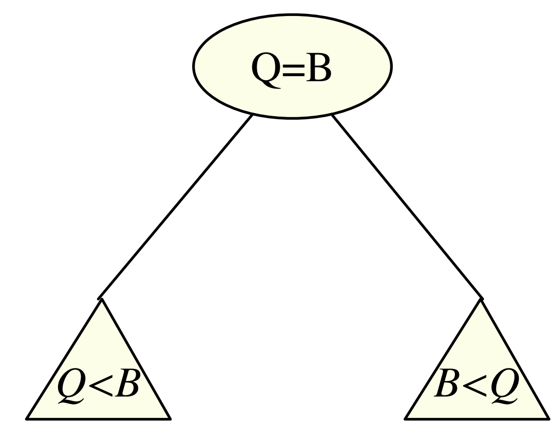
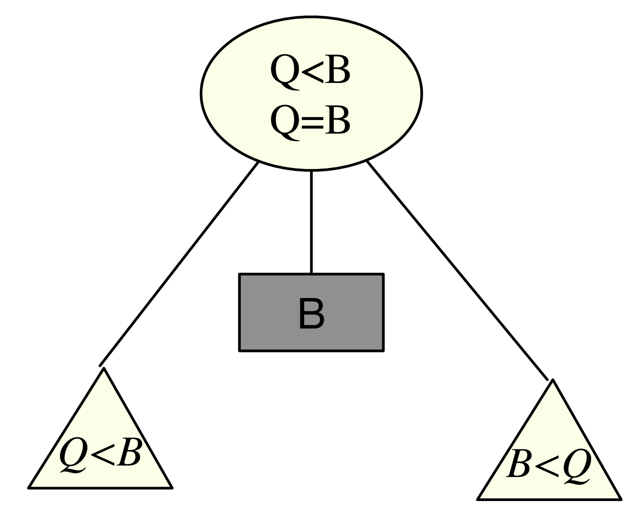


- In a *binary **comparison** search tree,* each internal node performs only one comparison. Searches all terminate at leaves.



- First such trees constructed by Hu-Tucker, also in 1971. *O(n log n)*

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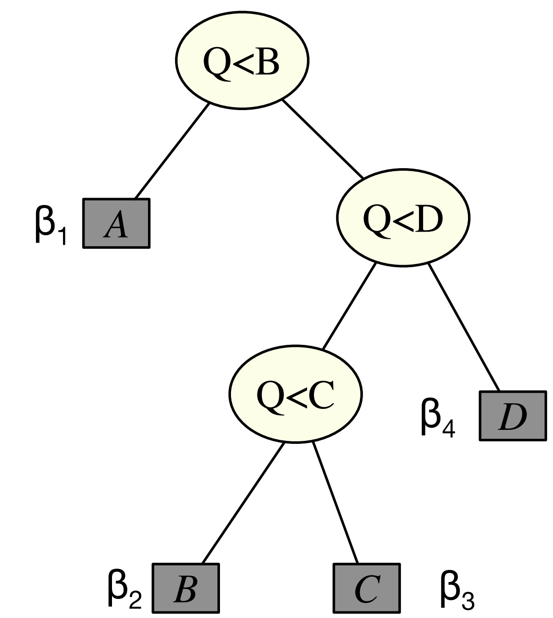

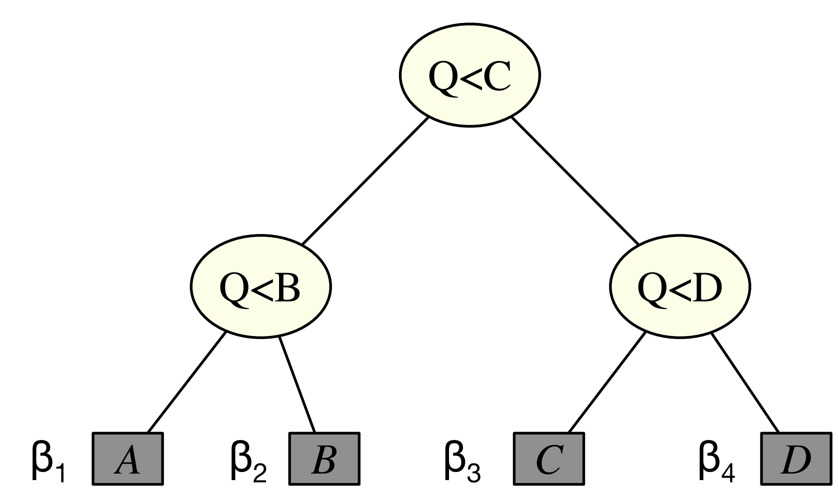

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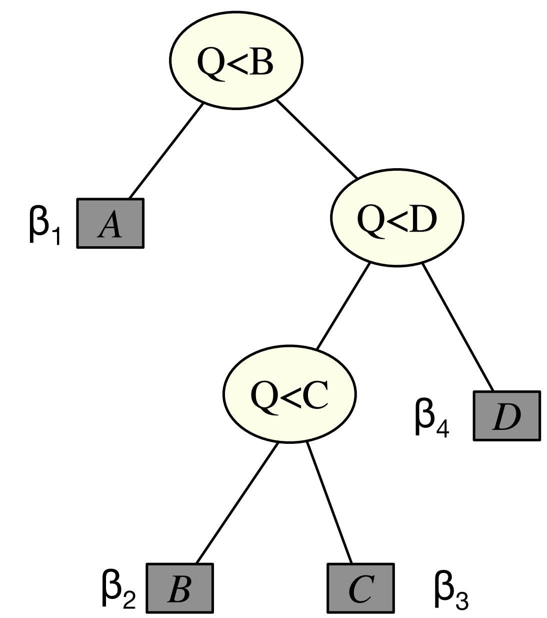

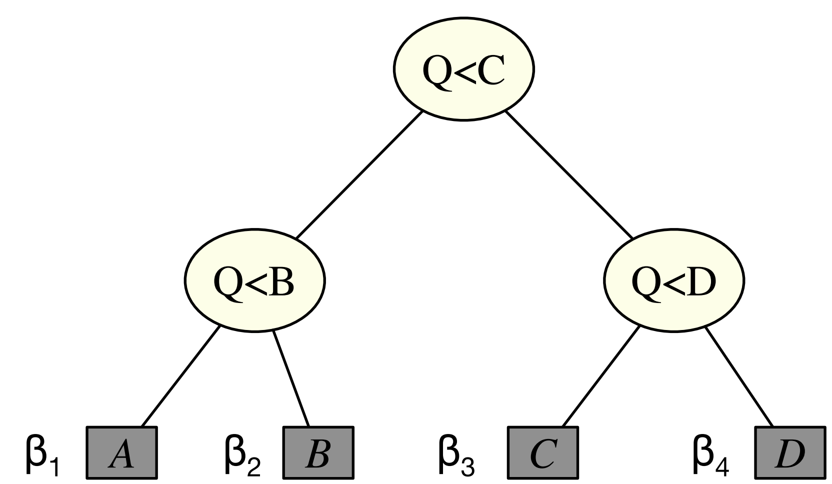

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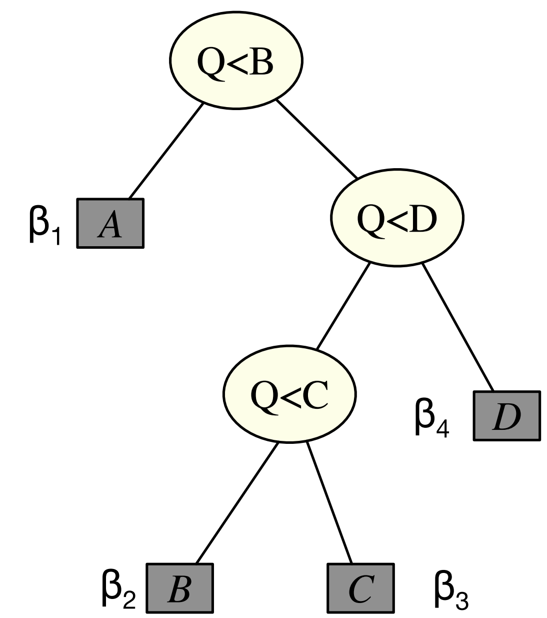

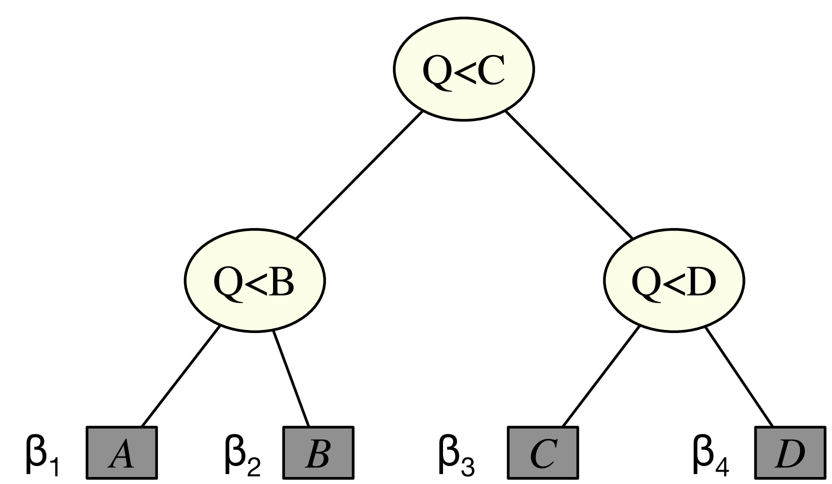

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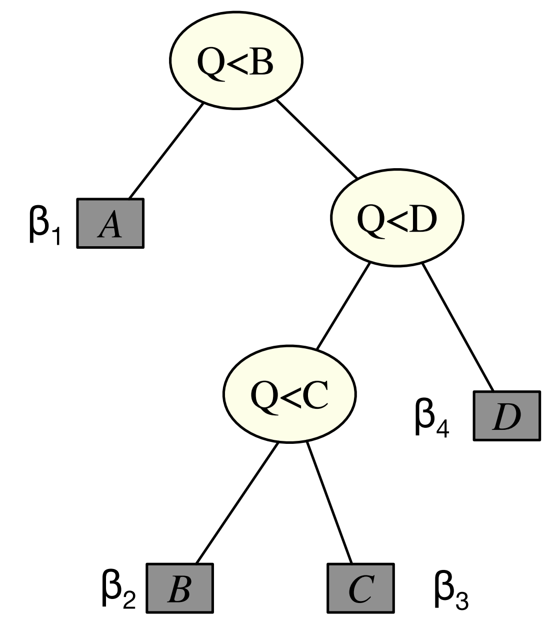

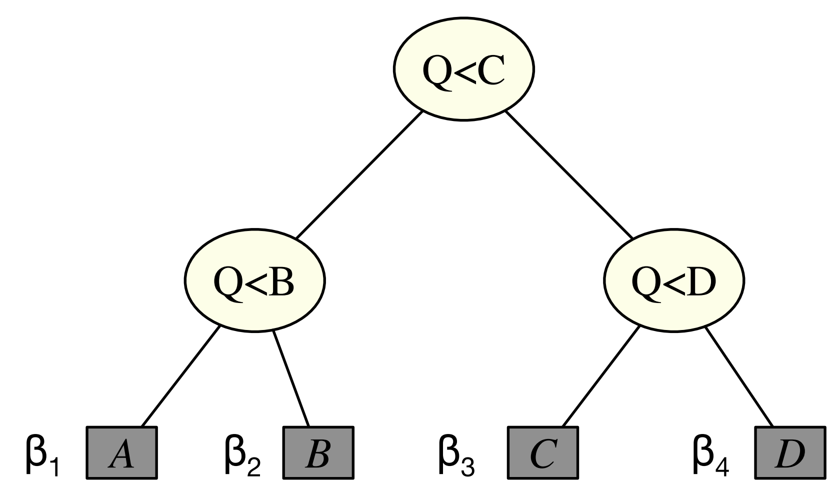

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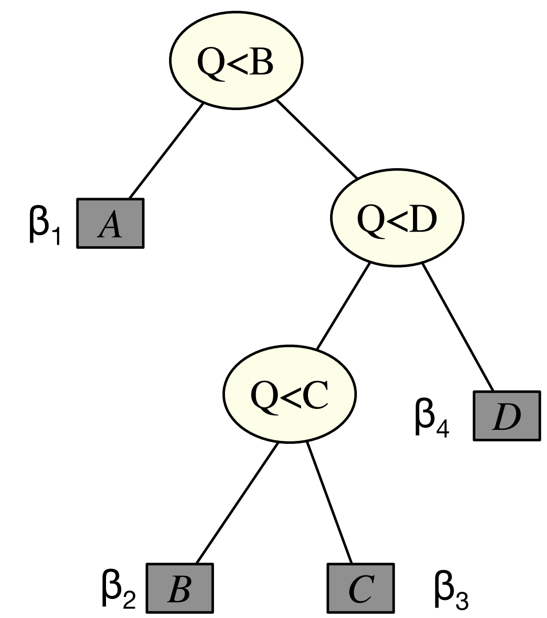

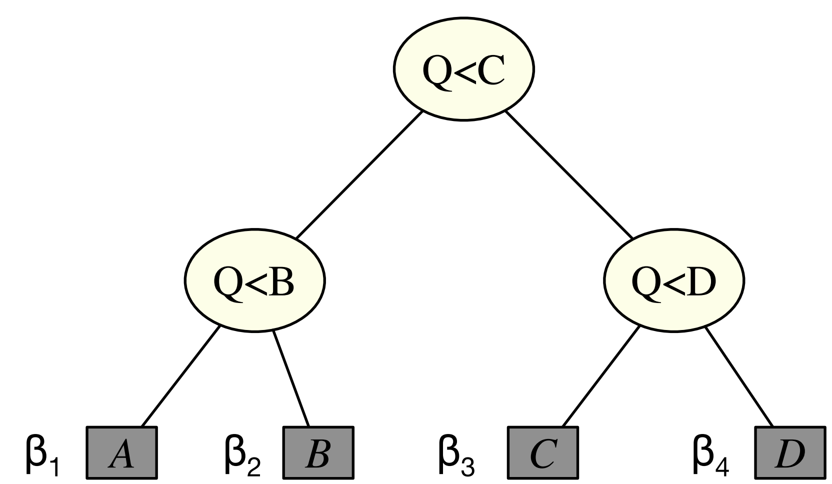

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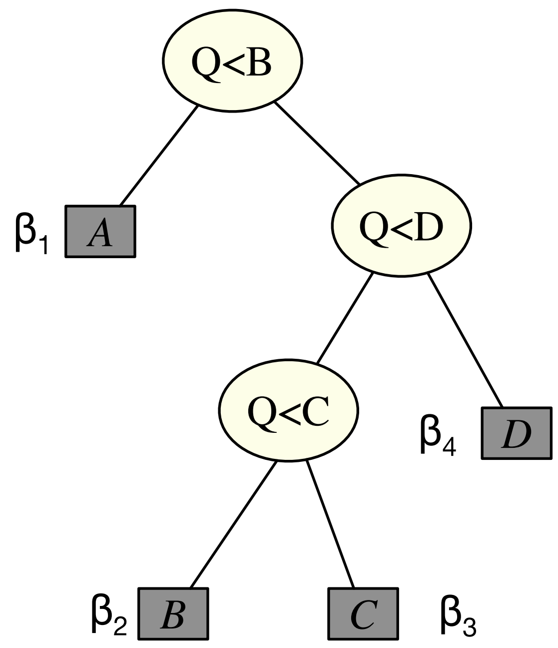

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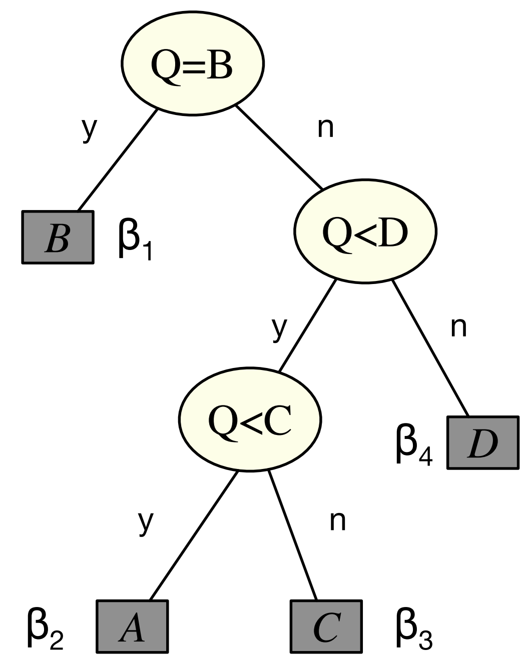

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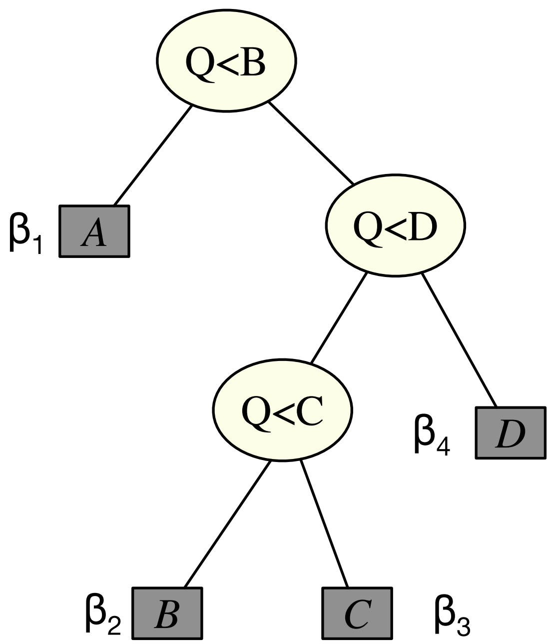

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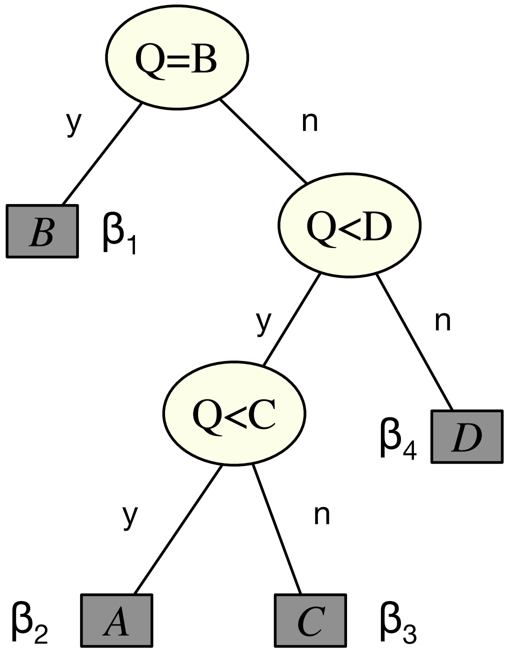

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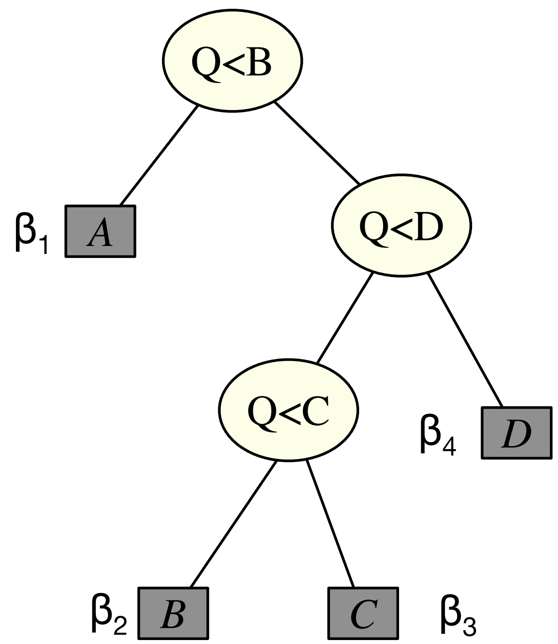

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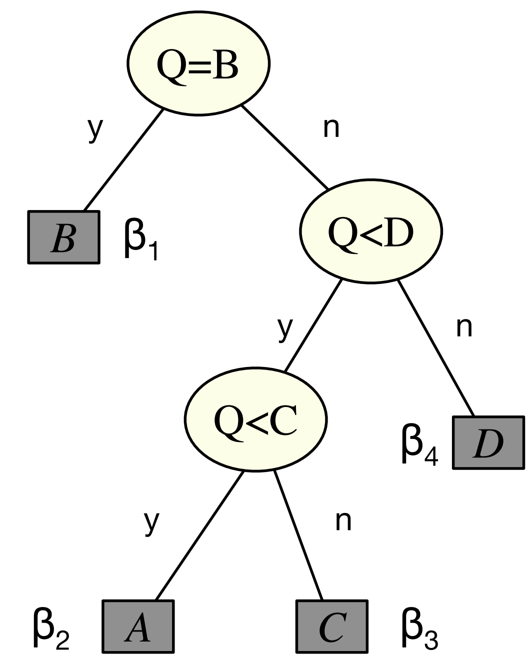

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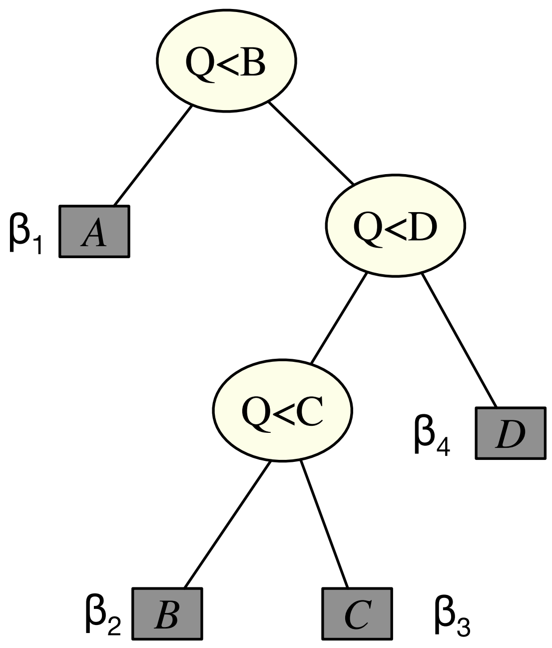

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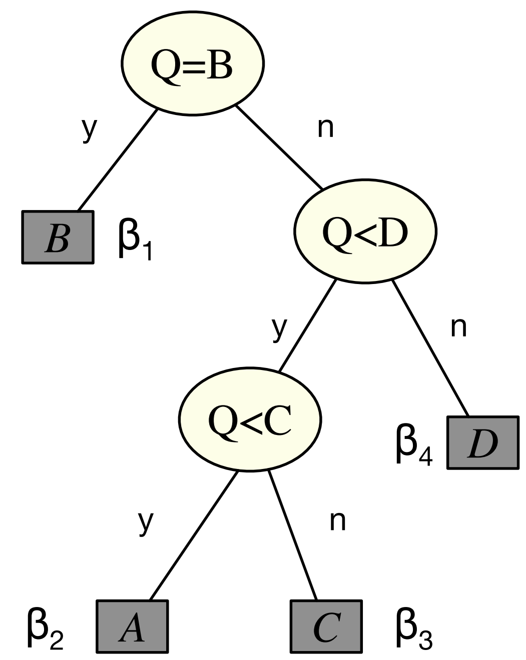

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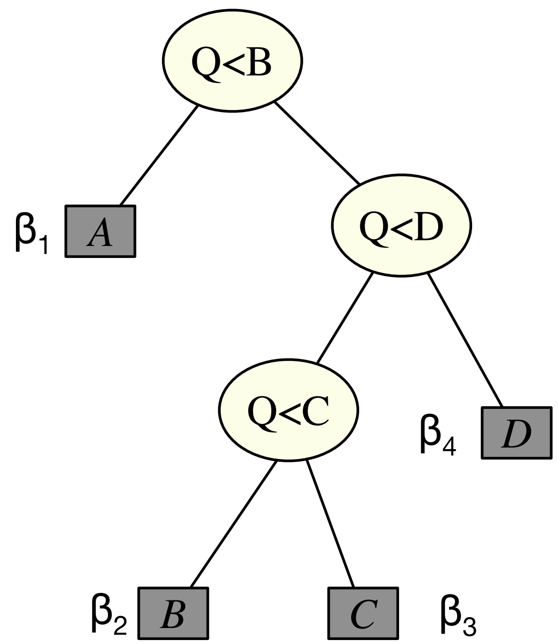
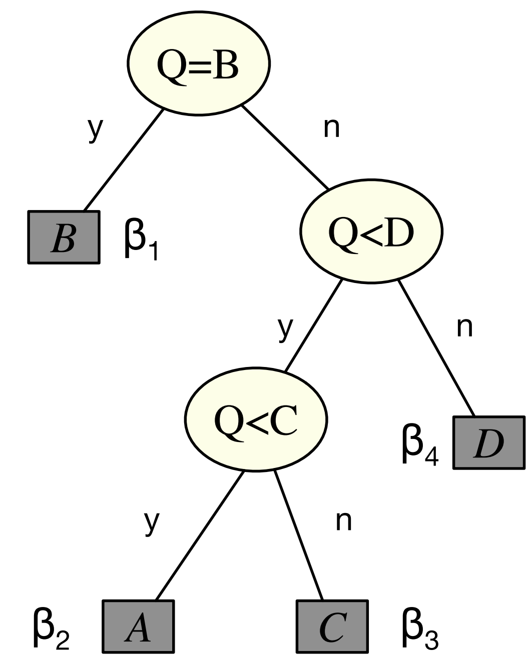
  - AKKL show that if equality comparison exists, then it is always largest probability in range. Allows recovering DP approach with ranges of description size $O(n^3)$ (compared to Knuth's $O(n^2)$)

Adding Equality Comparisons: AKKL[2001]



Hu-Tucker Tree



AKKL Tree

- *Comment 1 : Other problem in AKKL is how to deal with repeated weights This was hardest part.*
- *Comment 2: Both Hu-Tucker and AKKL only work when failures don't occur. I.e., only β_i are allowed and not α_i.*

So Far + Obvious Open Problem

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- Optimal Binary Search Trees
  - Input: $\beta_i = \Pr(Q = K_i);\ \alpha_i = \Pr(K_{i-1} < Q < K_i)$
  - *O(n²)* Knuth

- Optimal Binary Comparison Search Trees
  - Input: $\beta_i = \Pr(Q = K_i);\quad \text{failures not allowed}$
  - C = {<}: *O(n logn)* Hu-Tucker & Garsia-Wachs
  - C = {=,<}: *O(n⁴)* AKKL

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- Obvious Questions
  - Can we build OBCSTs that allow failures?
    - If yes, for which sets of comparisons?
  - Answer is yes, (for all sets of comparisons) but first need to define problem models

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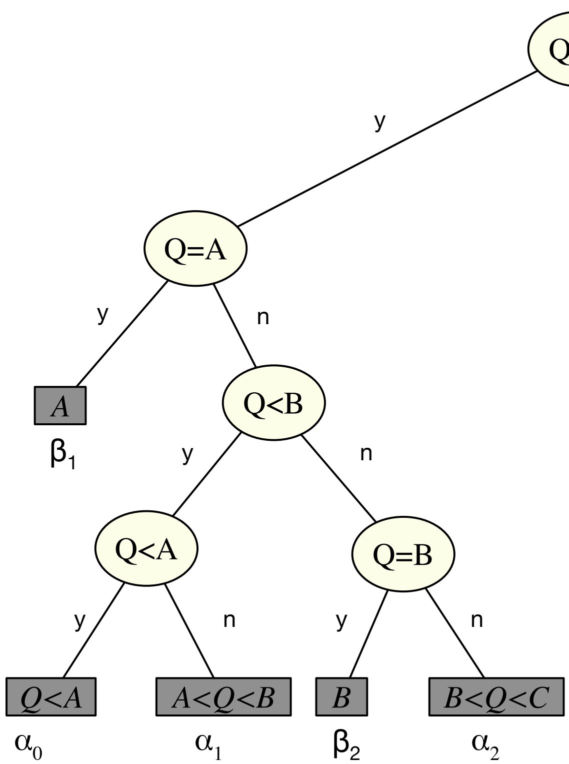
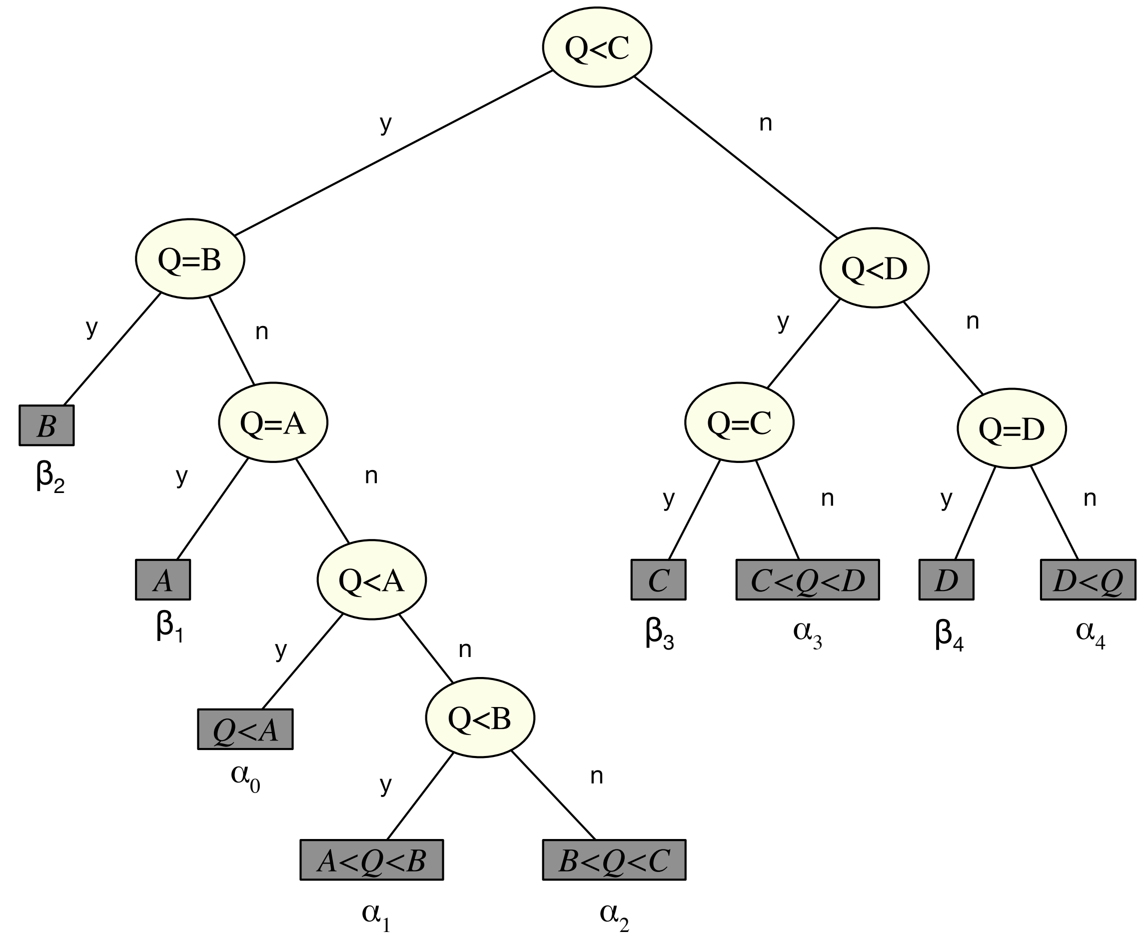
BCSTs with Failure Probabilities





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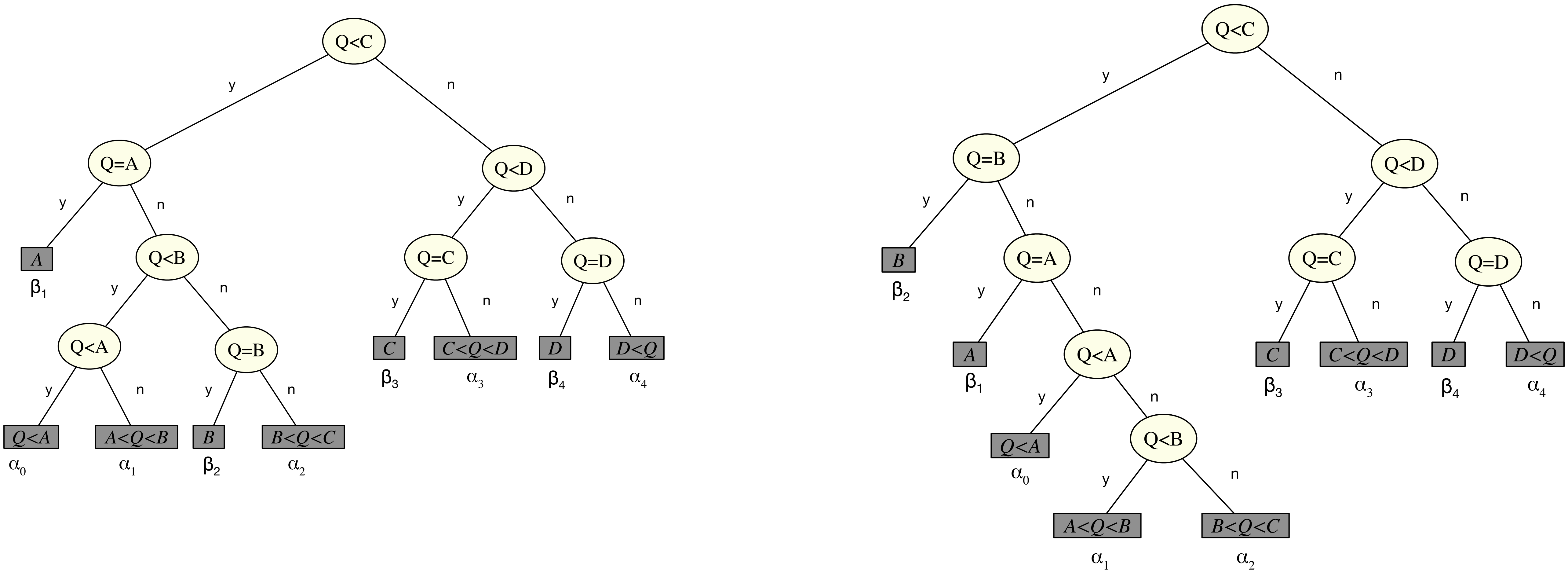
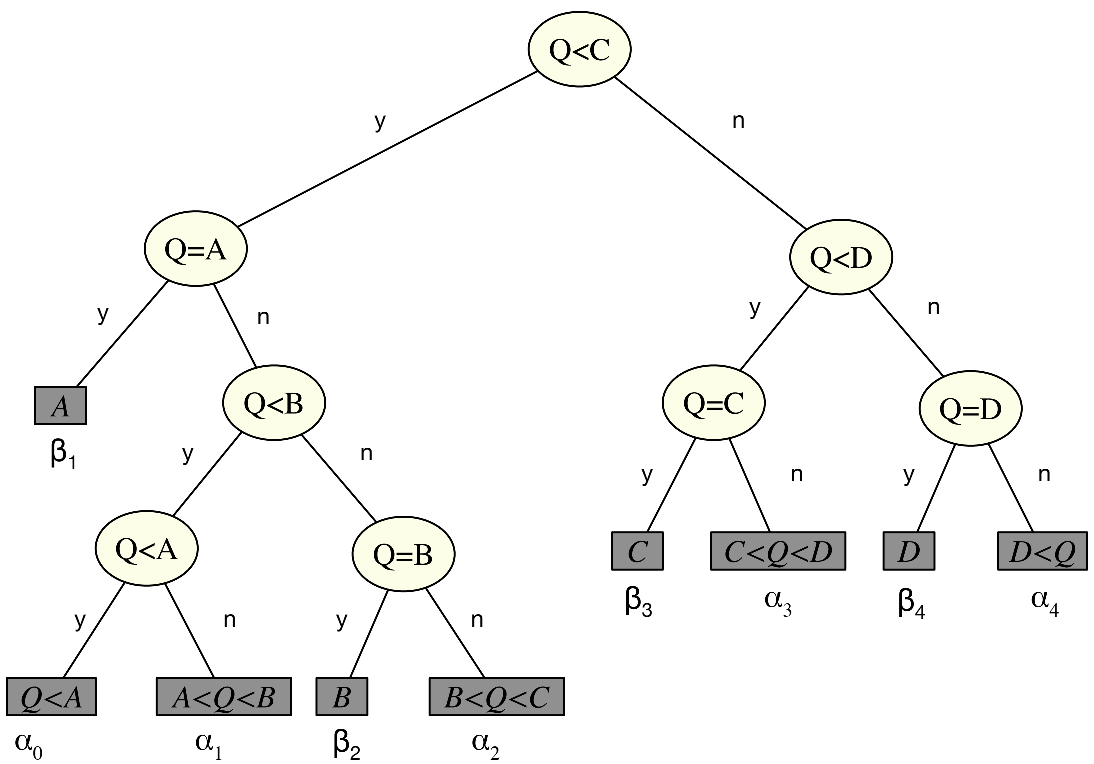
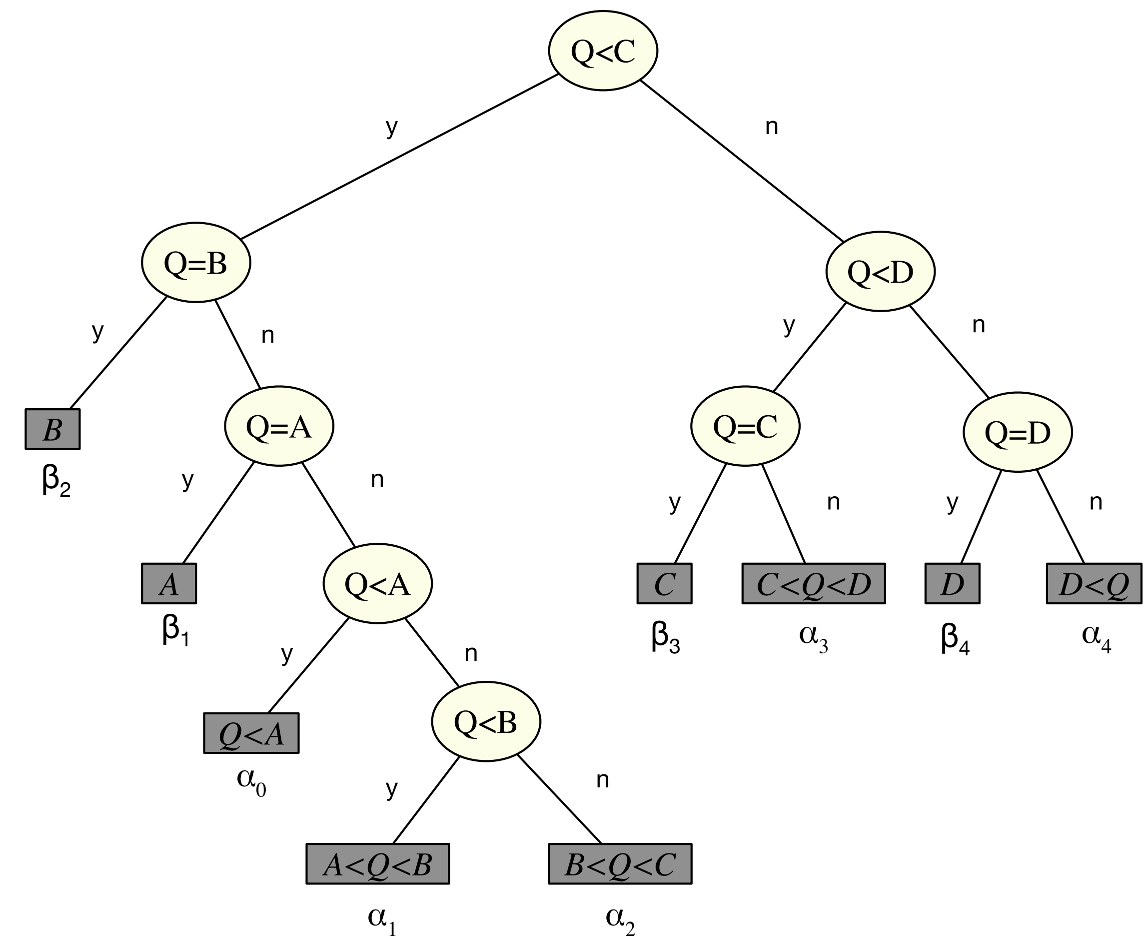


- Allows Failures (β_i and α_i).
  - Call this *complete input*. HT has *restricted* input.

BCSTs with Failure Probabilities





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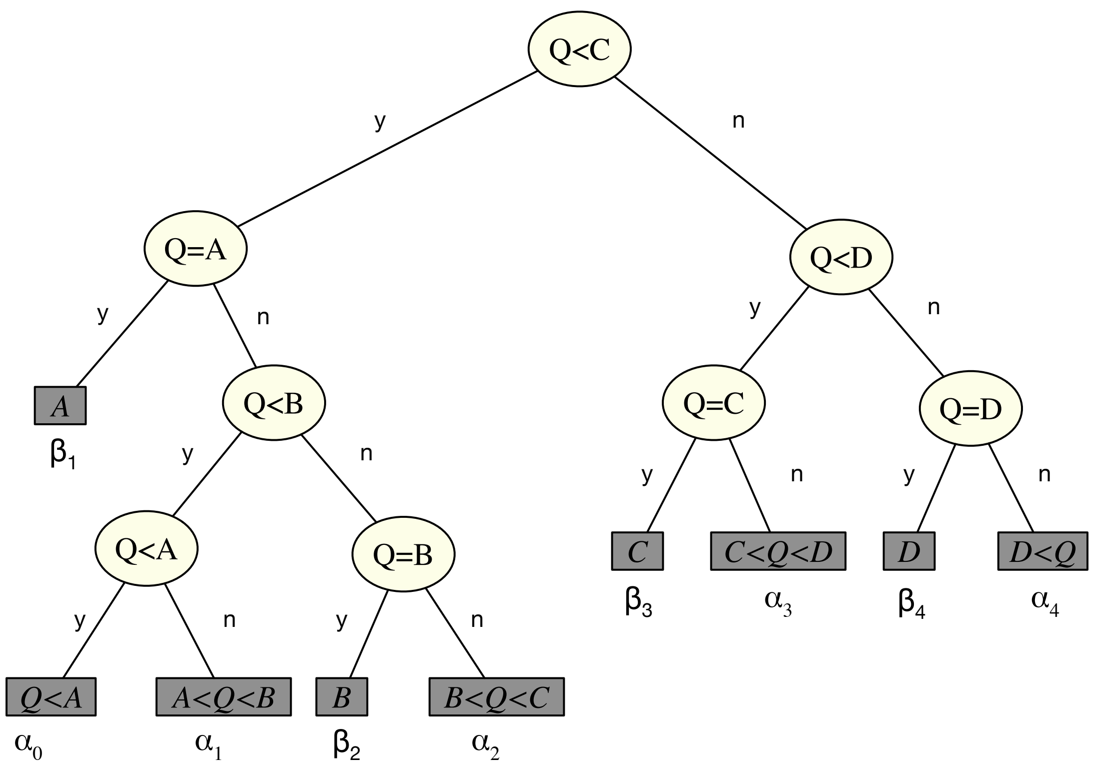

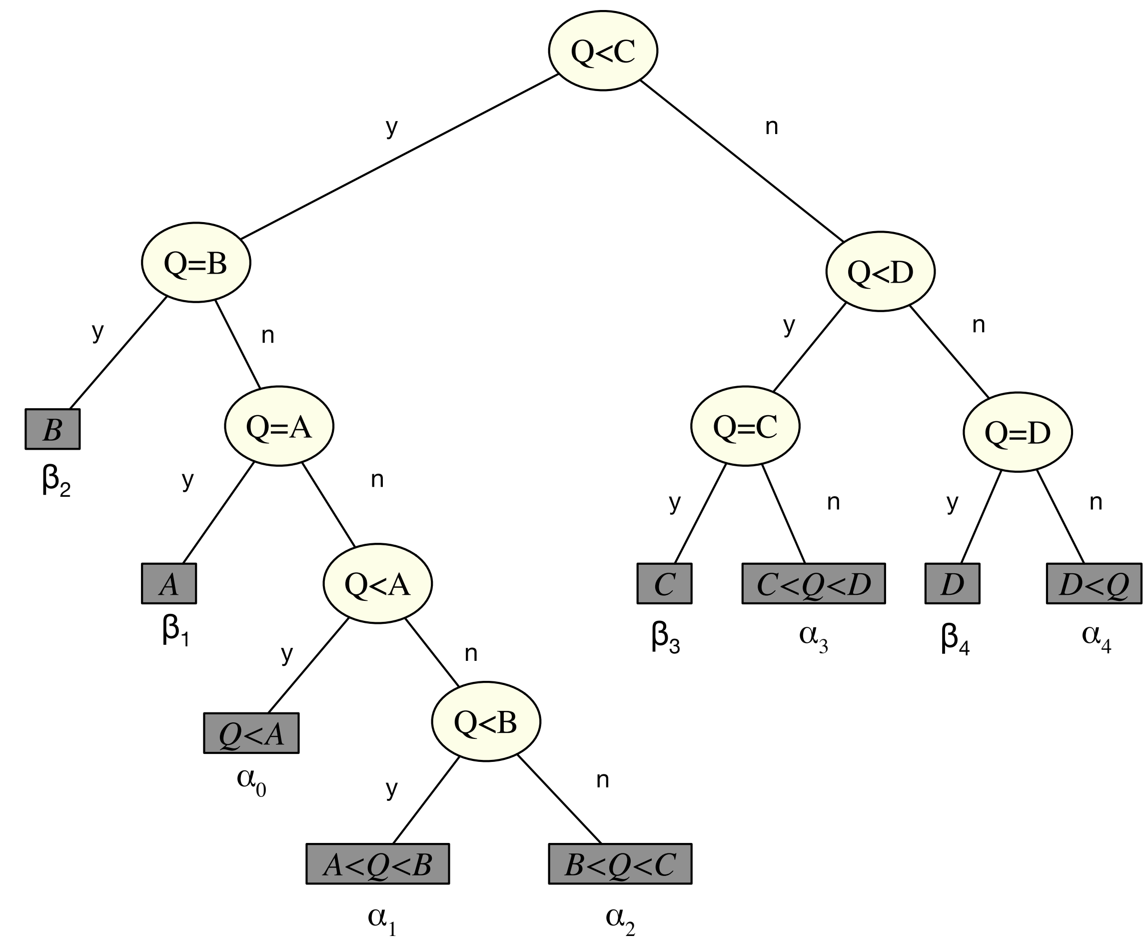

- Allows Failures (β_i and α_i).
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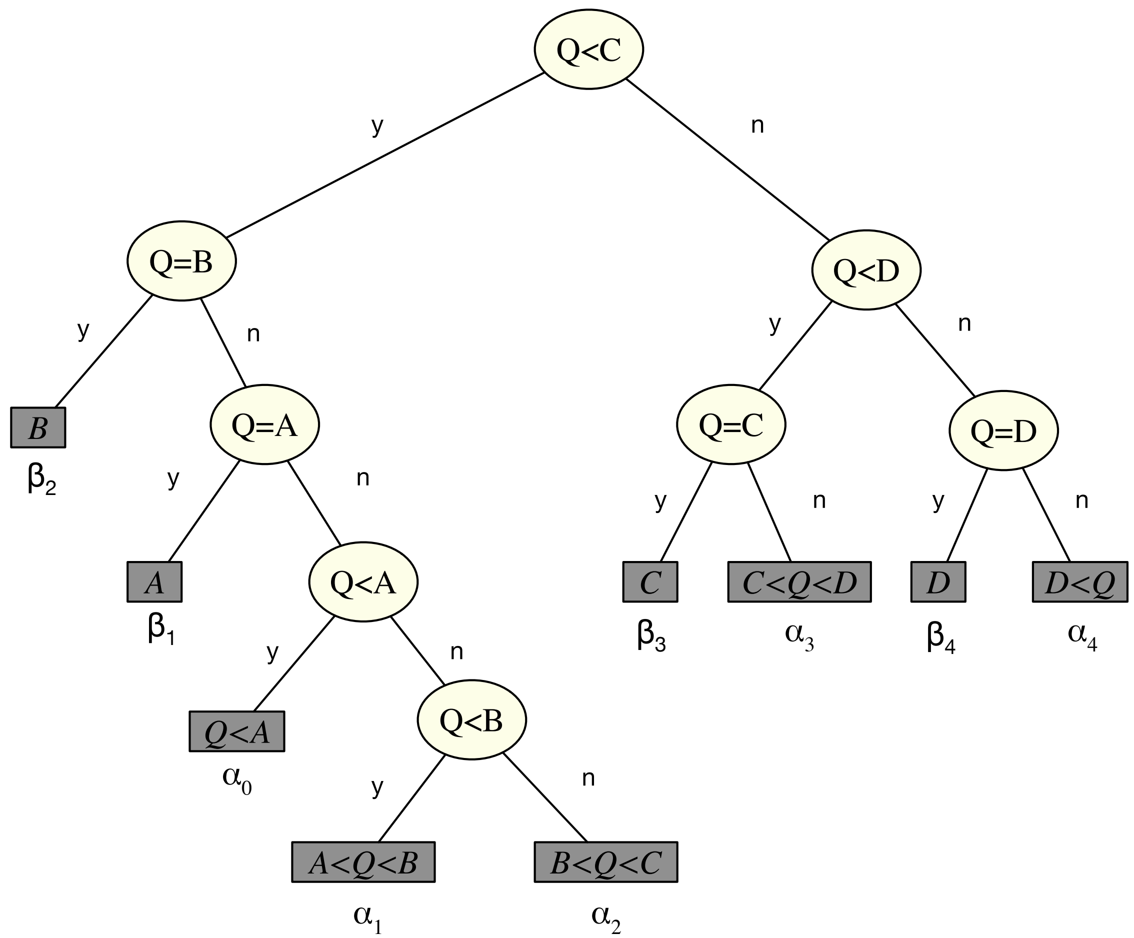
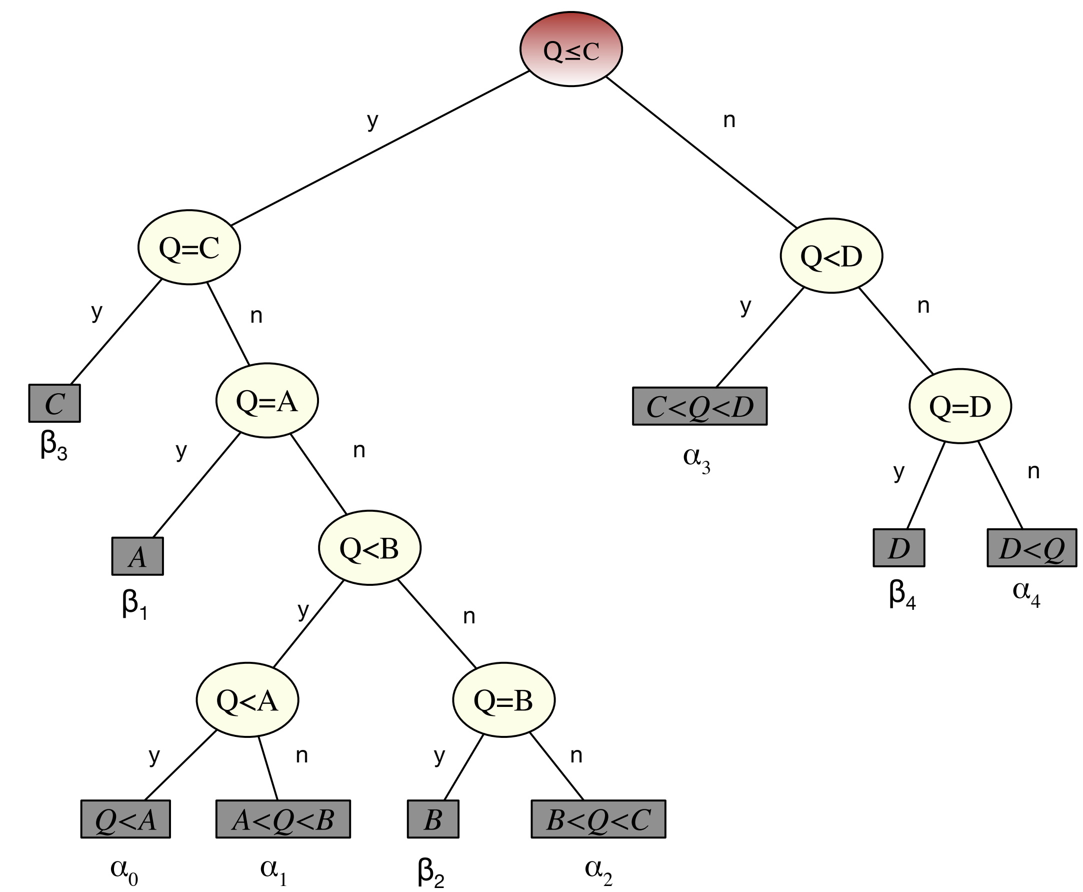
- Tree for *n* keys has *2n+1* leaves
- Distinguishing between $Q{=}{=}K_i$ and $K_i < Q < K_{i+1}$ always requires querying ($Q=K_i$)

Using Different Types of Comparisons





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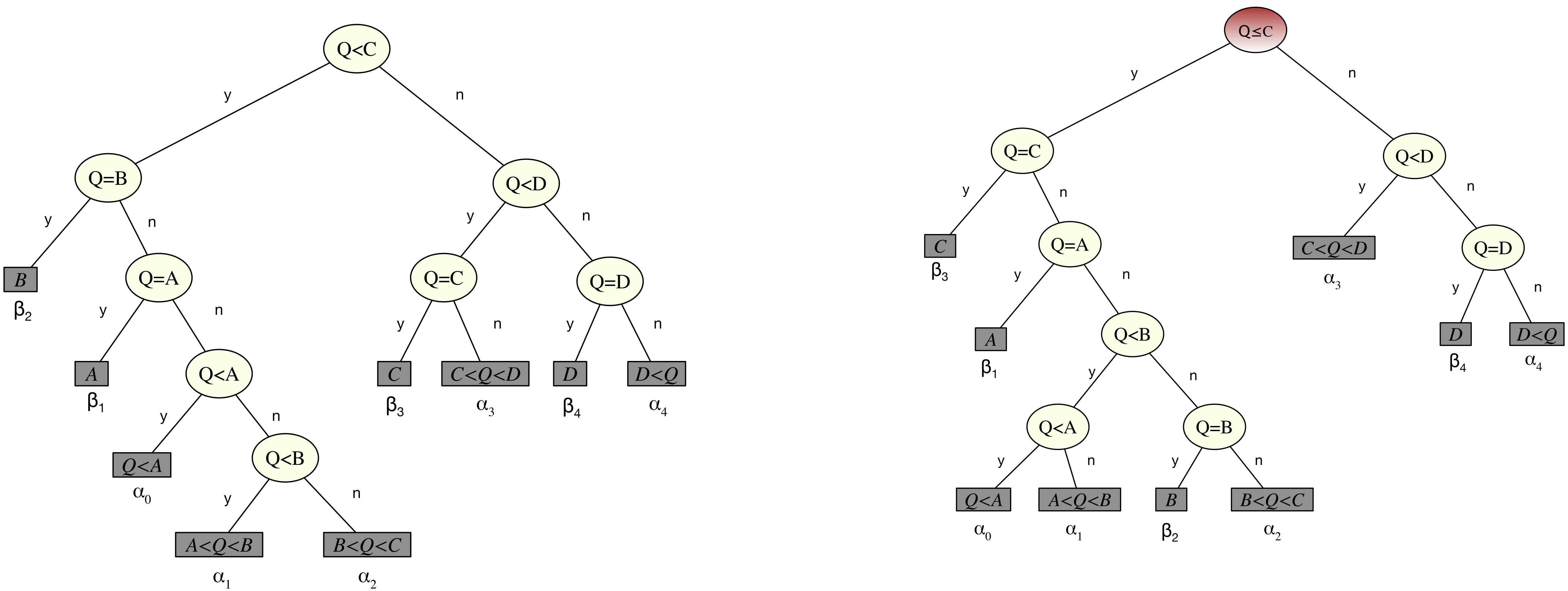

- Left Tree uses {<,=}. Right Tree uses {<, ≤, =}
  - Minimum cost BCST is minimum taken over all trees using given set of comparisons *C, e.g.*, C={<,=} or C={<, ≤, =}

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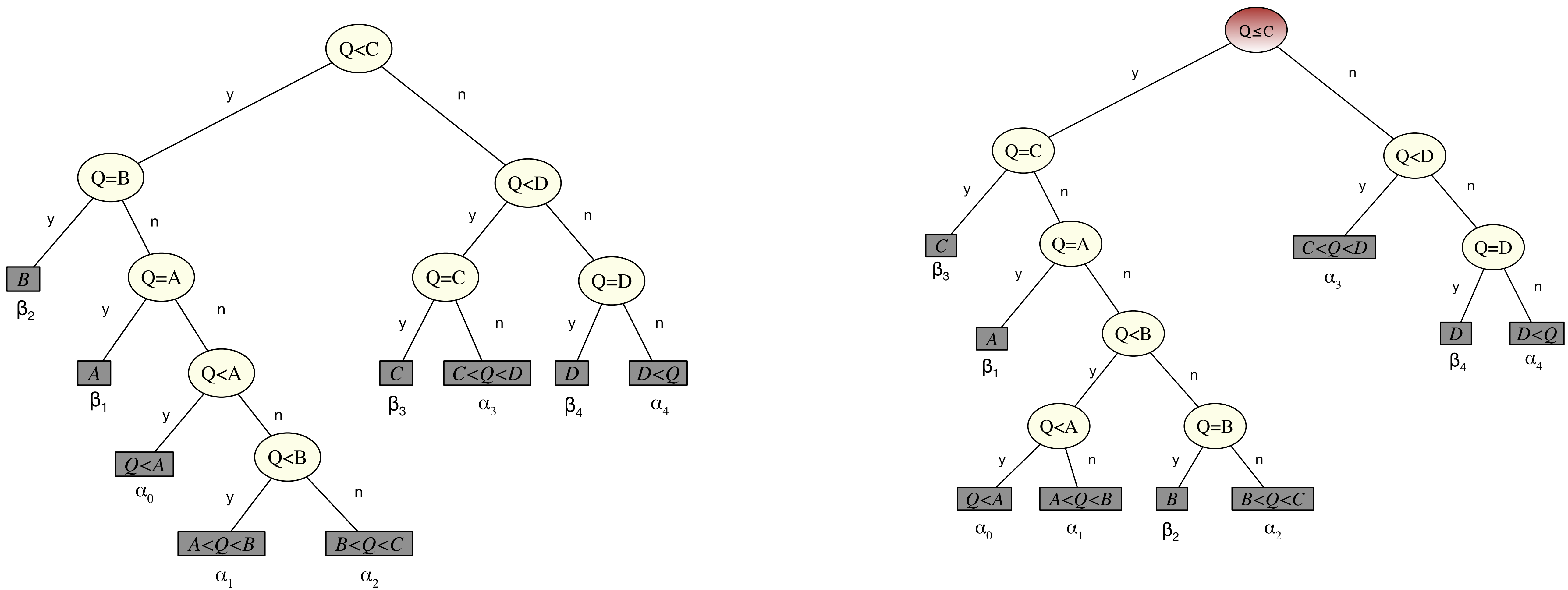
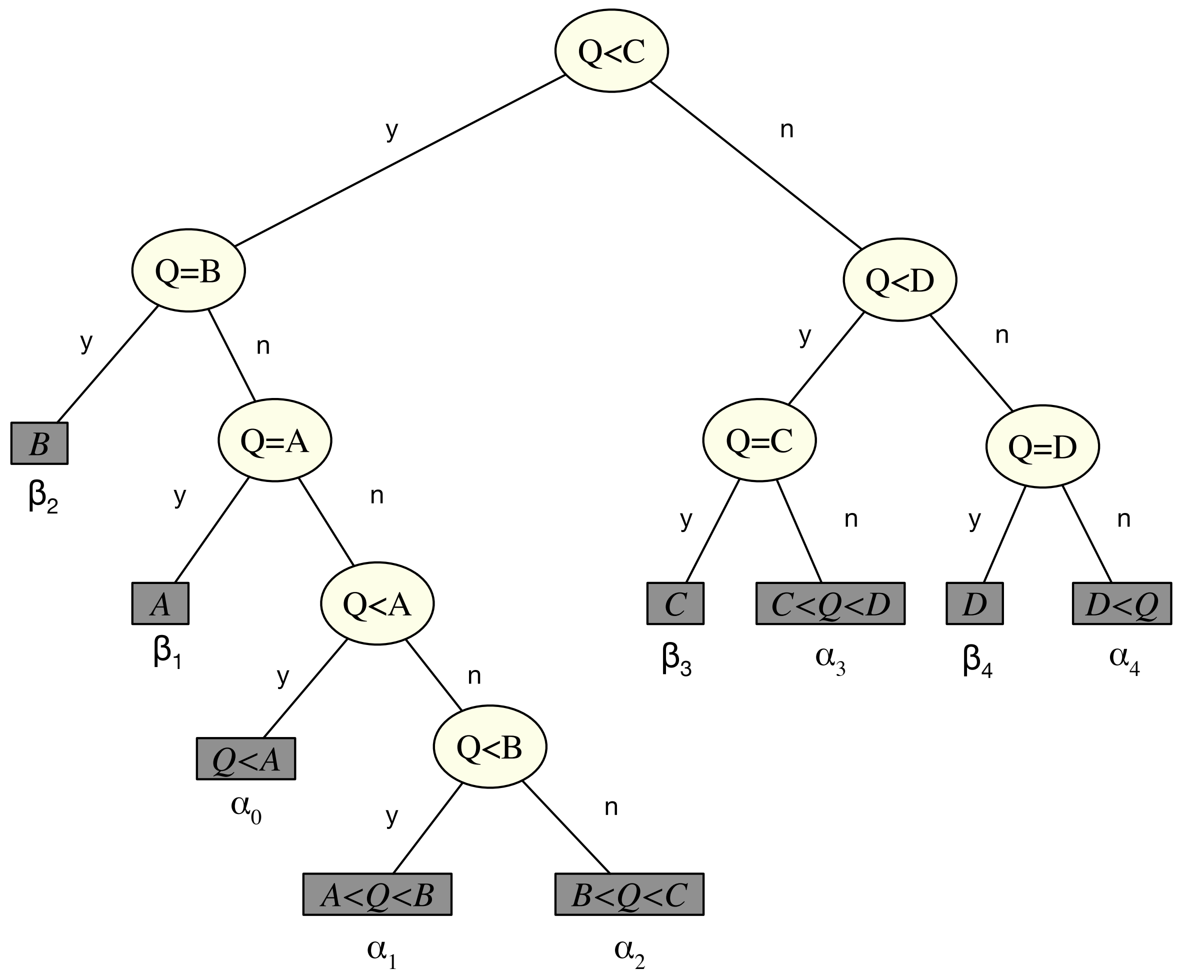
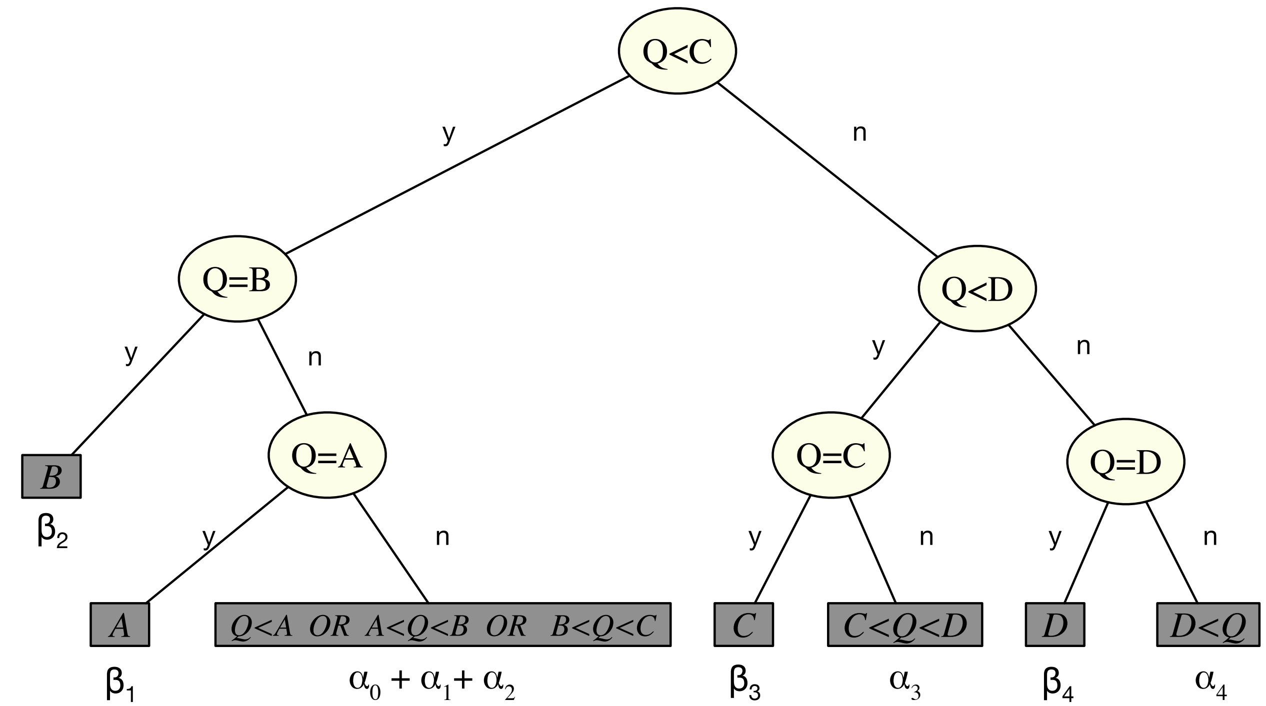


- Left Tree uses {<,=}. Right Tree uses {<, ≤, =}
  - Minimum cost BCST is minimum taken over all trees using given set of comparisons *C, e.g.,* C={<,=} or C={<, ≤, =}
- *C* is input to the problem.
  - Algorithm is different for different *C*s.

How Much Information is Needed for Failure?





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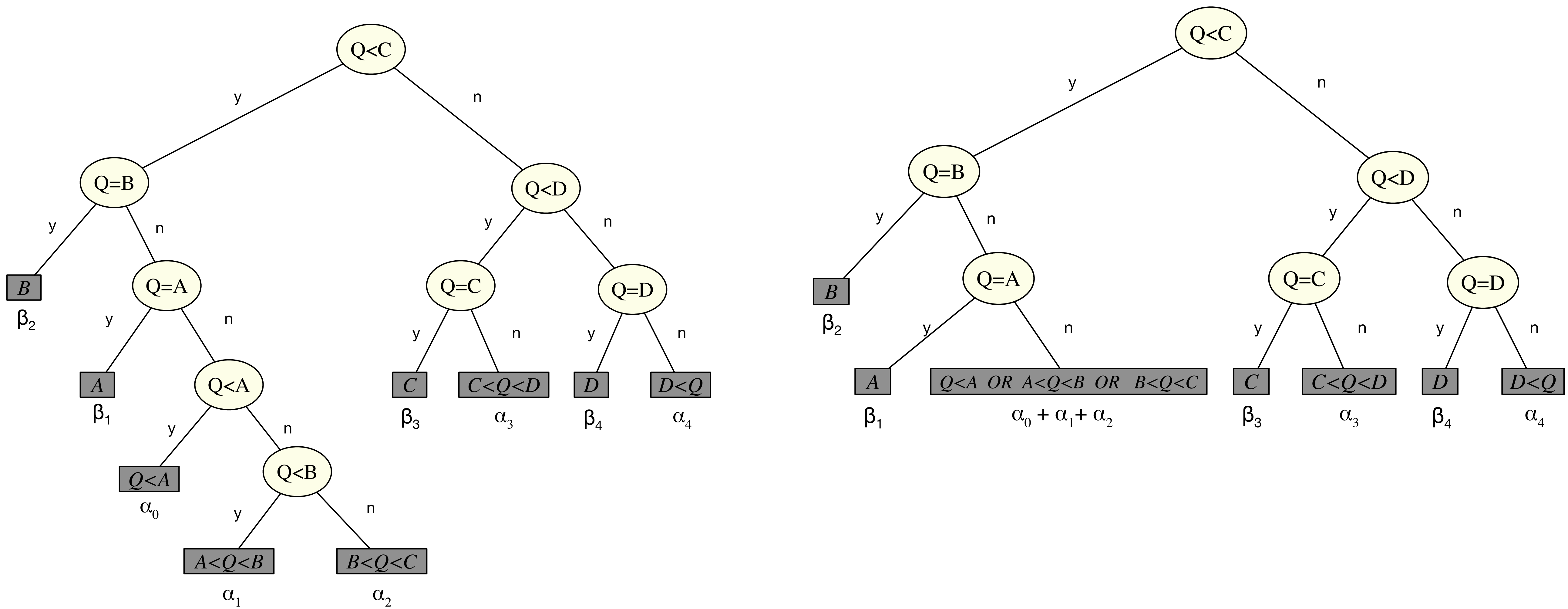

- Tree on left shows **Explicit Failure**
  - every failure leaf reports unique failure interval, $K_i < Q < K_{i+1}$.

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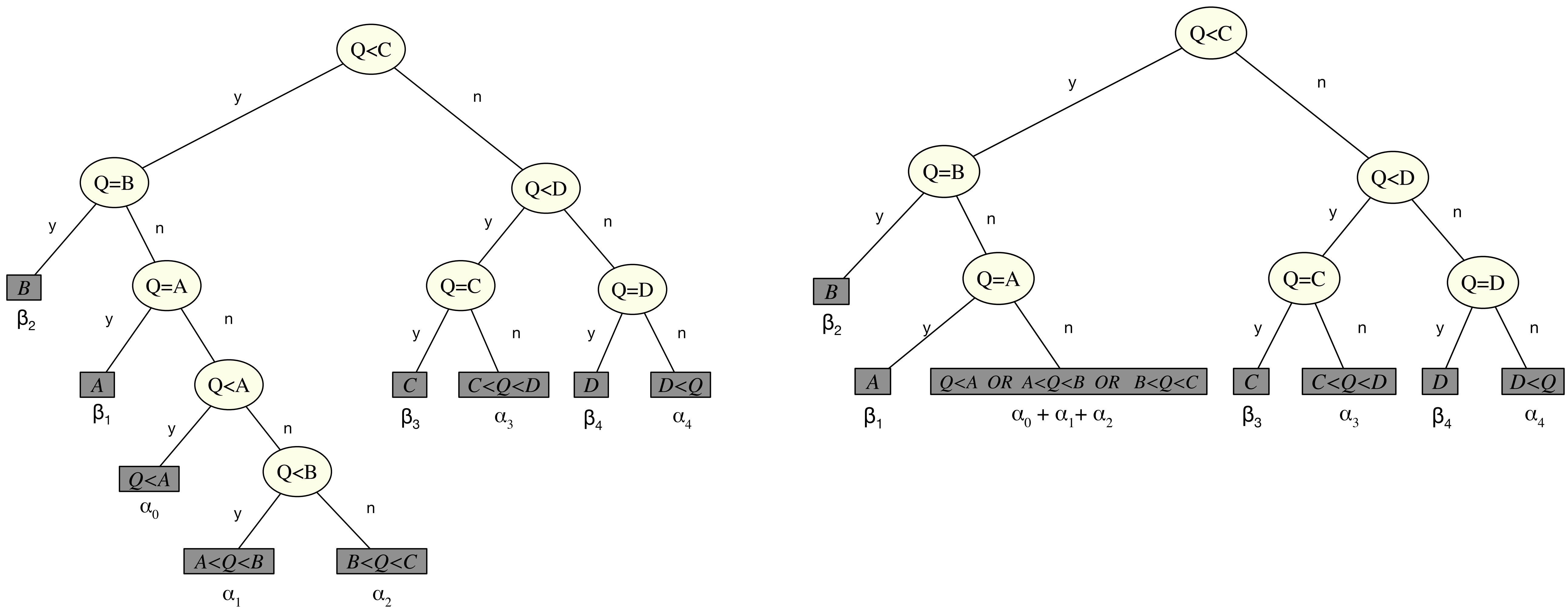

- Tree on left shows **Explicit Failure**
  - every failure leaf reports unique failure interval, $K_i < Q < K_{i+1}$.
- Tree on right shows **Non-Explicit Failure**:
  - Failure leaves only report failure. Don't need to specify exact interval. Leaf can be concatenation of successive failure intervals .

New Algorithms: OBCSTs with Failures

Permitted Comparisons	Failure Type	Time	Comments
$\mathcal{C} = \{=\}$	Explicit	—	Can not occur
	Non-Explicit	$O(n \log n)$	Trivial. Similar to Linked List
$\mathcal{C} = \{<, \leq\}$	Explicit	$O(n \log n)$	$O(n)$ Reduction to Hu-Tucker
	Non-Explicit	—	Can not occur
$\mathcal{C} = \{=, <\}$, $\mathcal{C} = \{=, \leq\}$	Explicit	$O(n^4)$	Follows from Main Lemma
	Non-Explicit	$O(n^4)$	"
$\mathcal{C} = \{=, <, \leq\}$	Explicit	$O(n^4)$	"
	Non-Explicit	$O(n^4)$	"

- DP Algorithms for last 4 cases are very similar
- Differ slightly in
  - Design of Recurrence Relations
    - {=,<} and {=,<, ≤) yield slightly different recurrences
  - Initial conditions
    - Explicit and Non-Explicit Failures force different I.C.s

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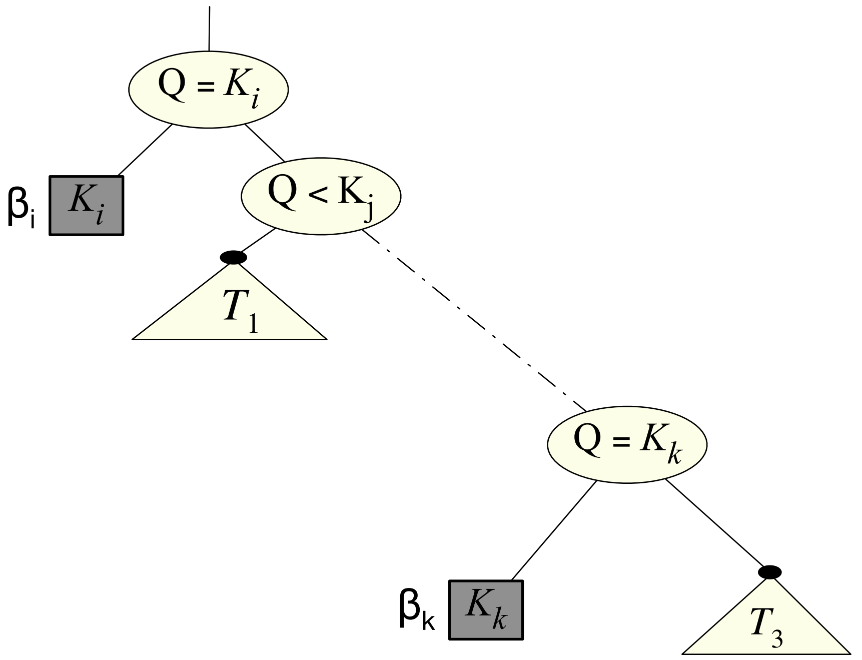
Main Lemma:

Lemma

Let T be a Optimal BCST.

If ($Q=K_k$) is a Descendant of ($Q=K_i$)

Then $\beta_k \leq \beta_i$



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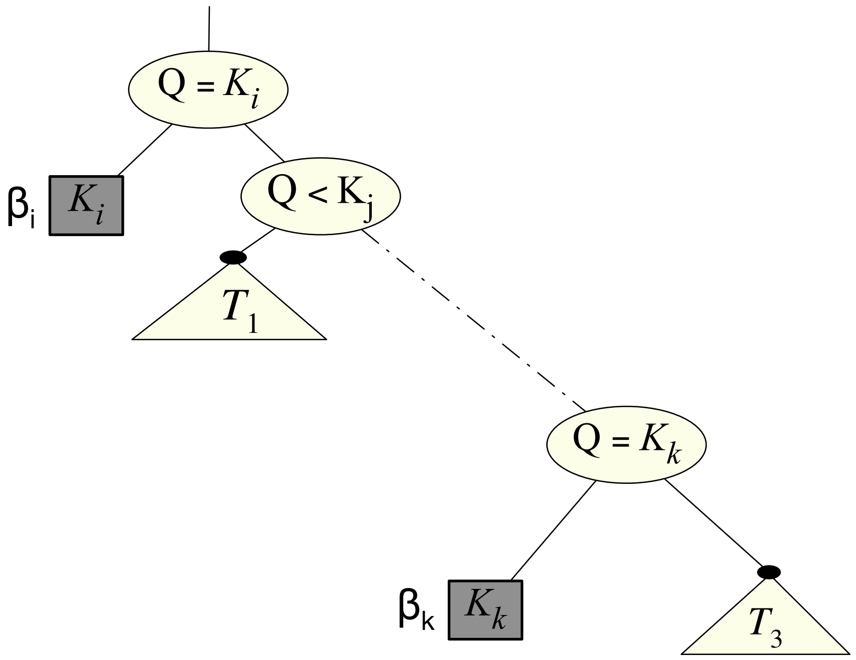

Note: This is true regardless of which inequality comparisons are used and which model BCST is used

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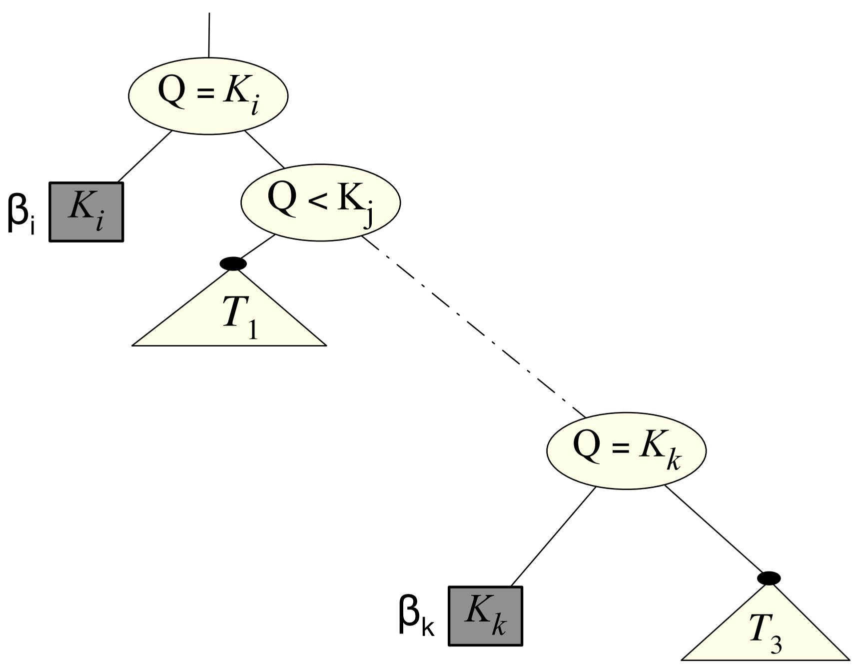

Note: This is true regardless of which inequality comparisons are used and which model BCST is used

Corollary: If T is an OBCST and ($Q=K_k$) an internal node in T, then $\beta_k \le \beta_j$ for all ($Q=K_j$) on the path from the root to ($Q=K_k$), i.e., equality weights decrease walking down the tree

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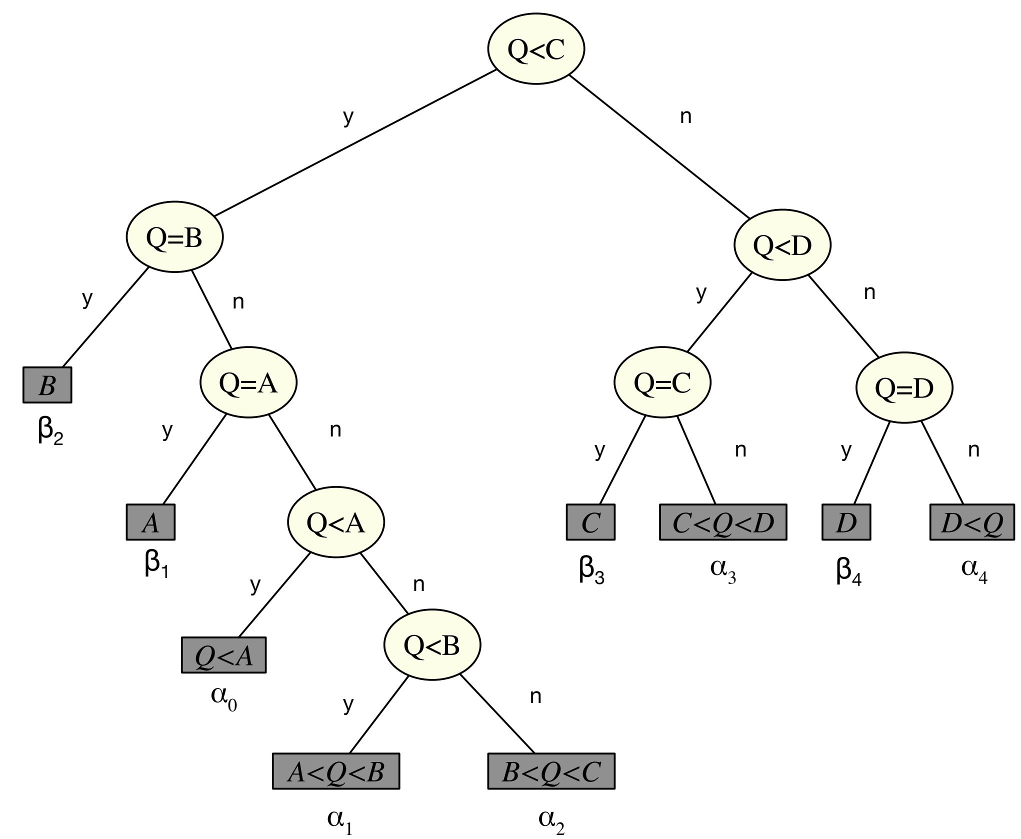
Structural Properties of BCSTs

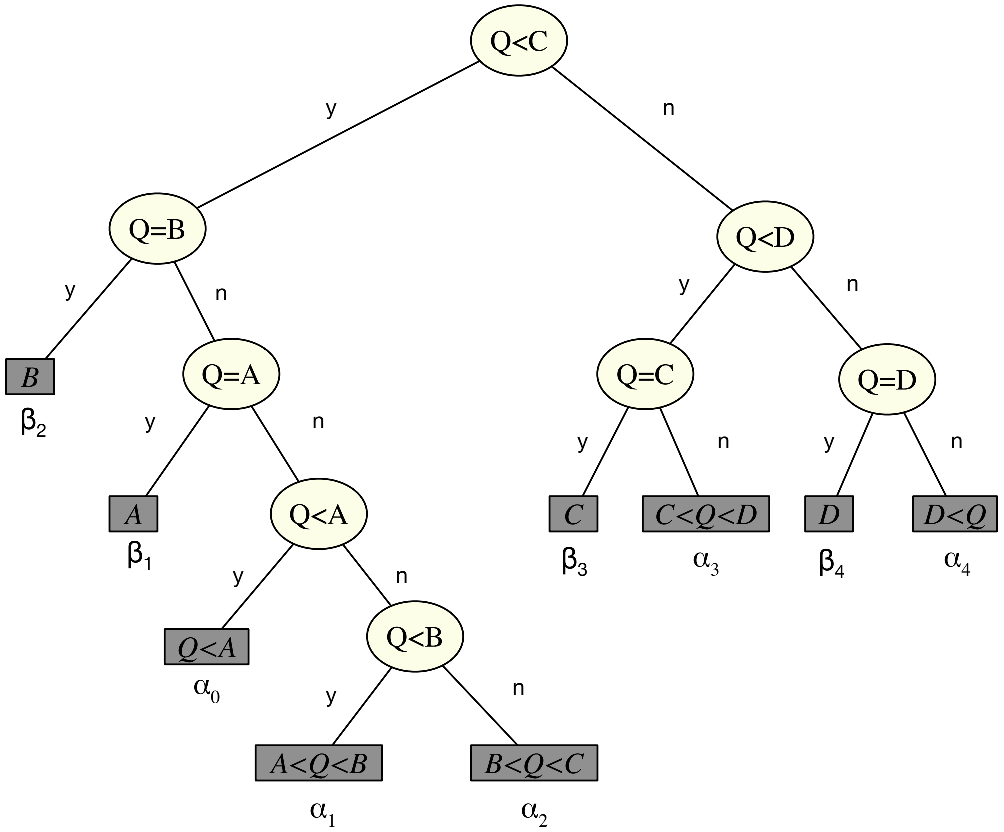


Structural Properties of BCSTs

Henceforth assume distinct key weights,

i.e., all of the $\beta_1, \beta_2, \ldots, \beta_n$ are different

Also assume C={<,=}



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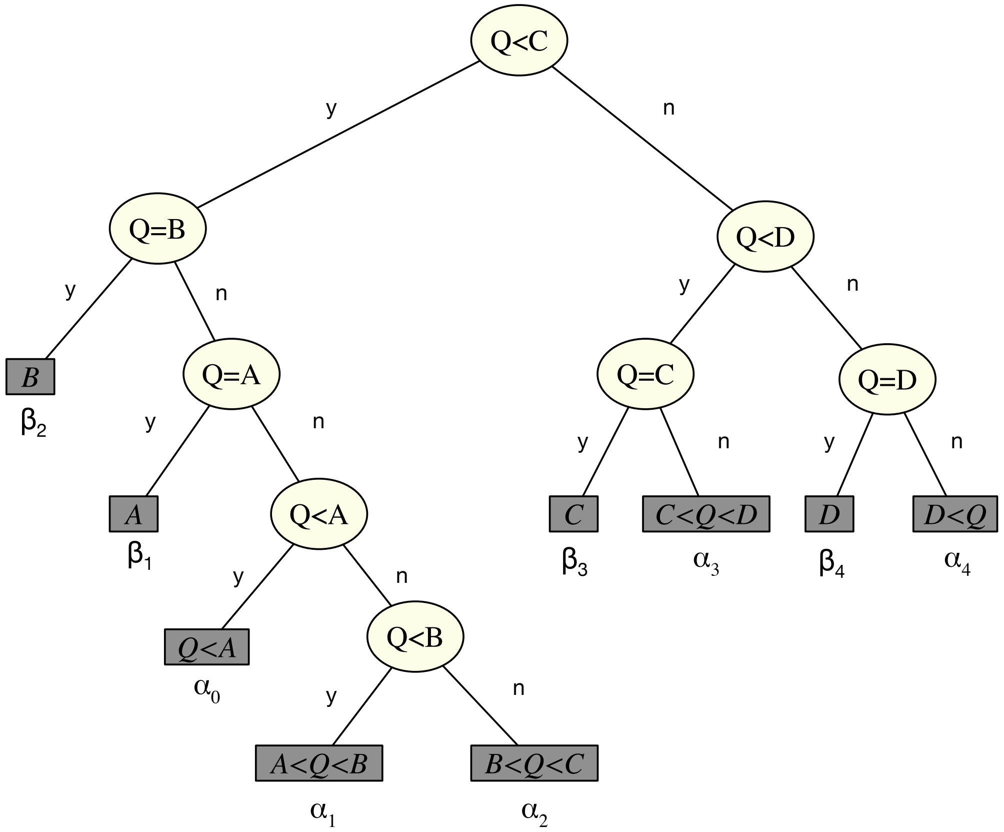

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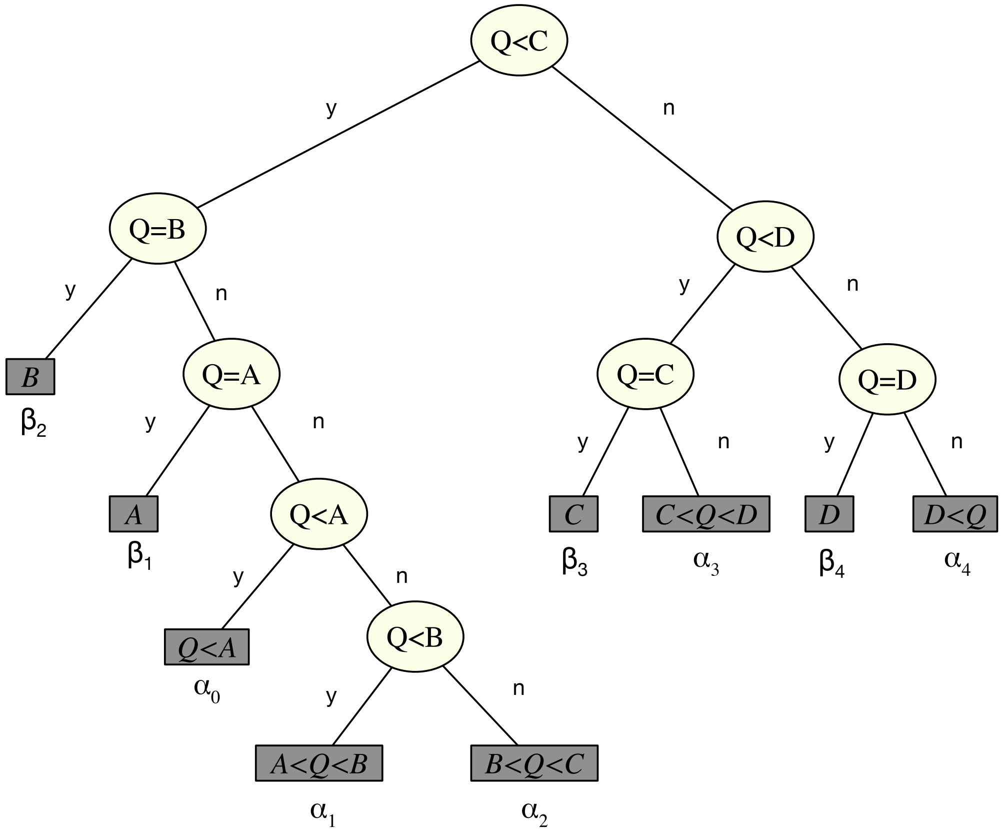

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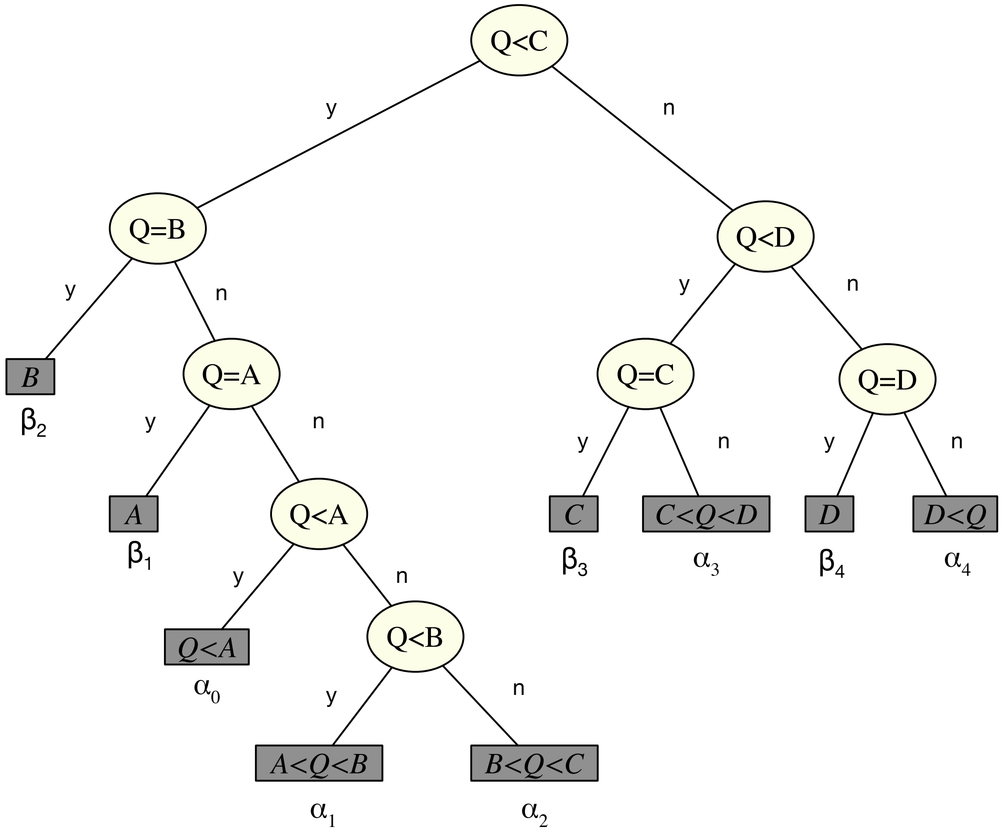

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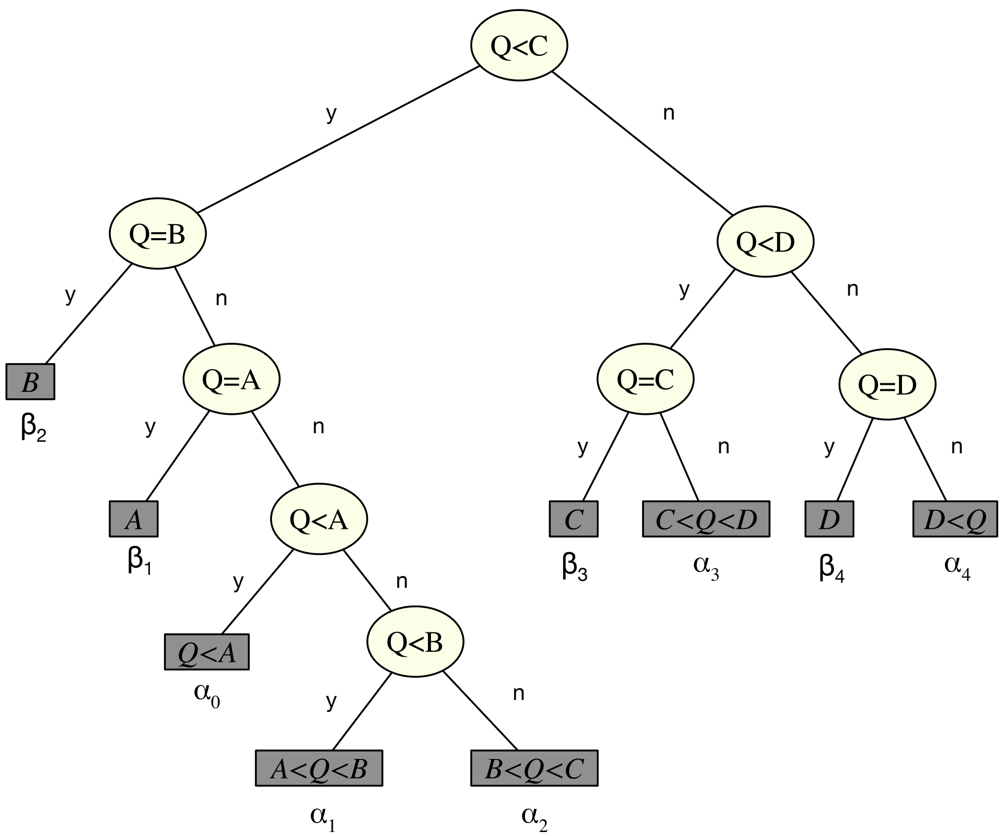

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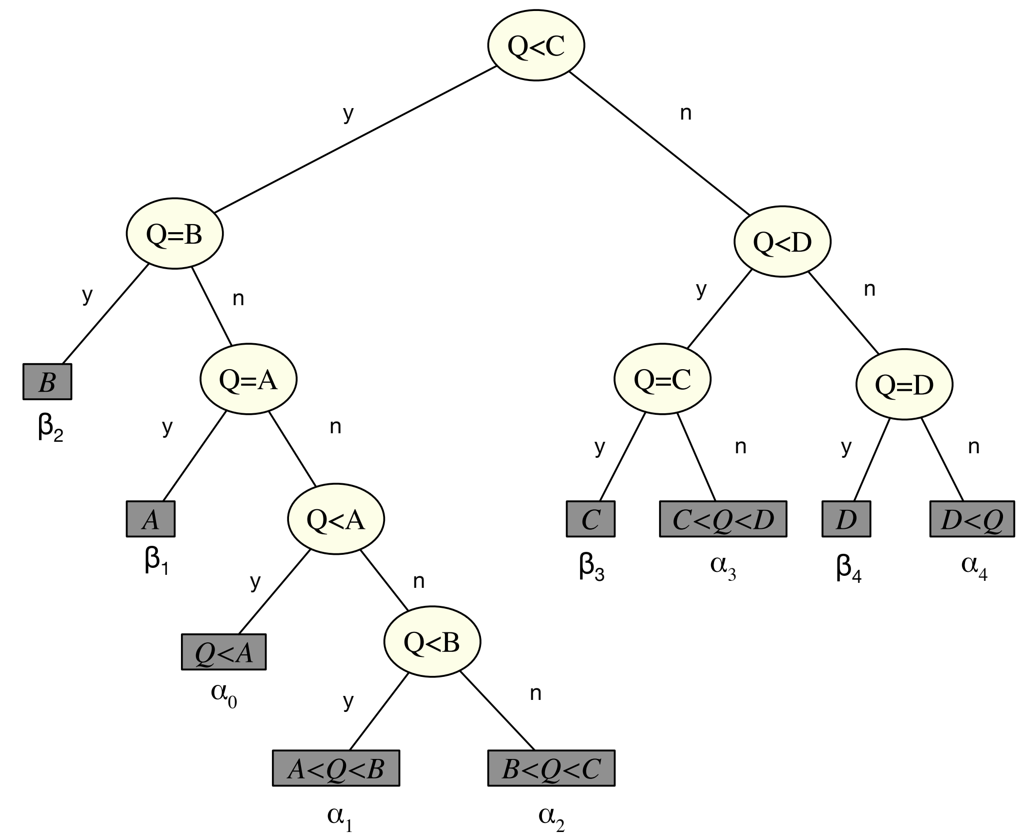

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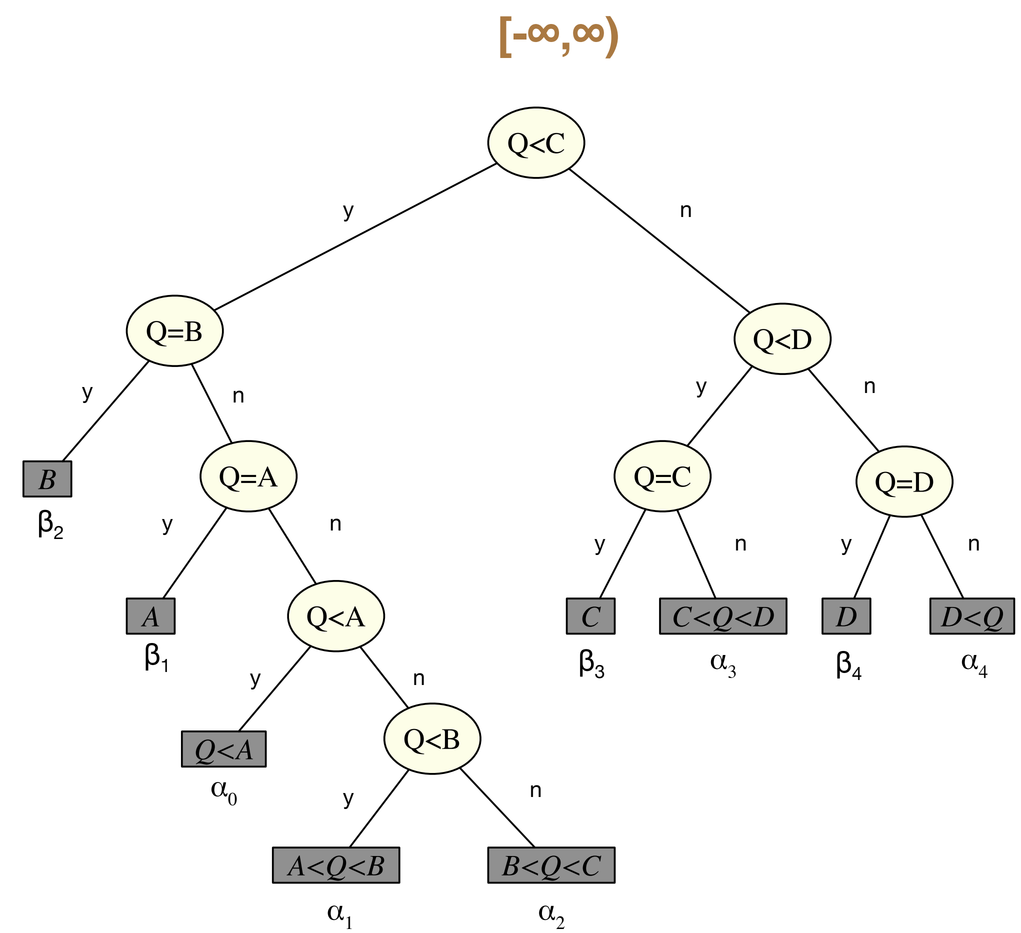

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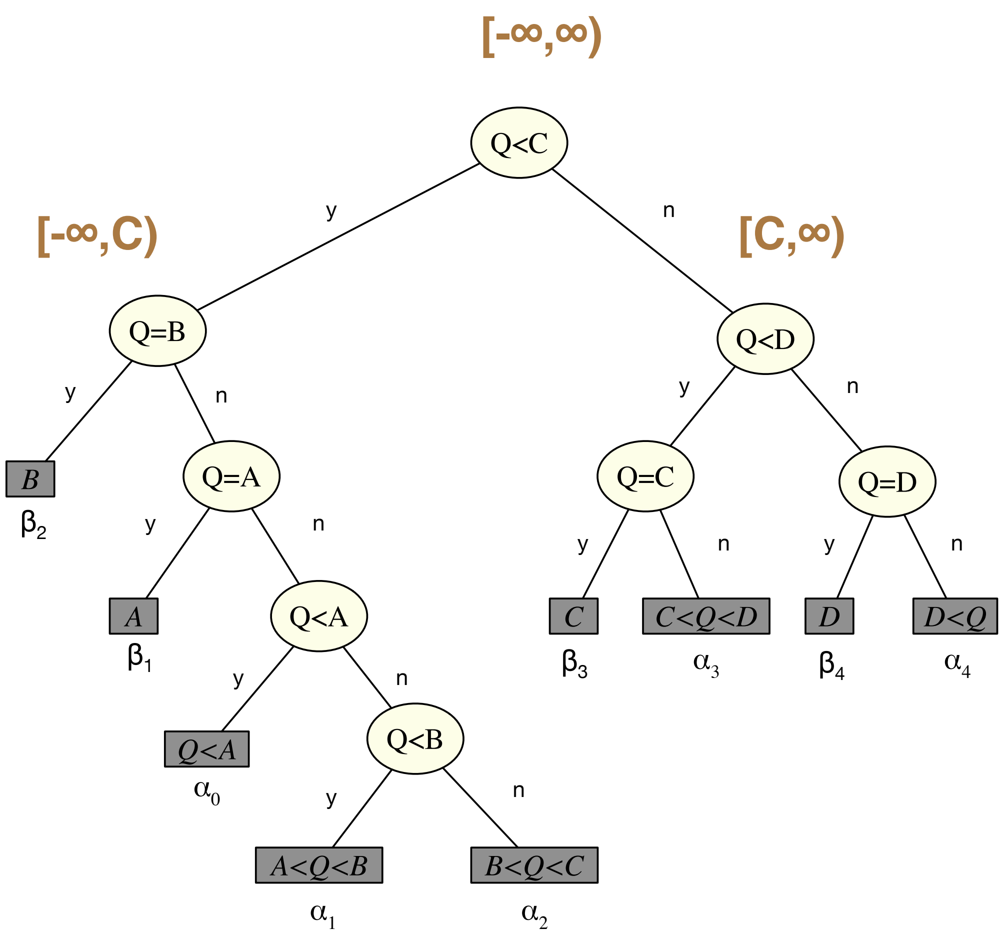

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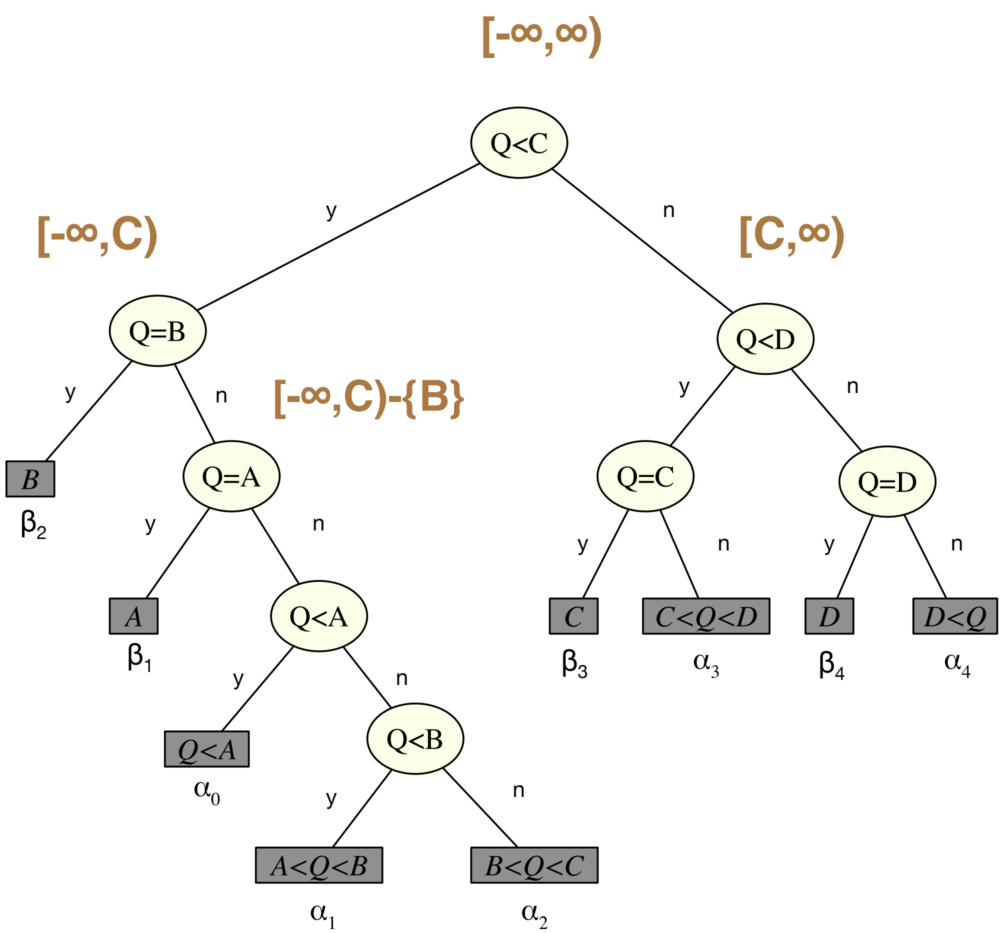

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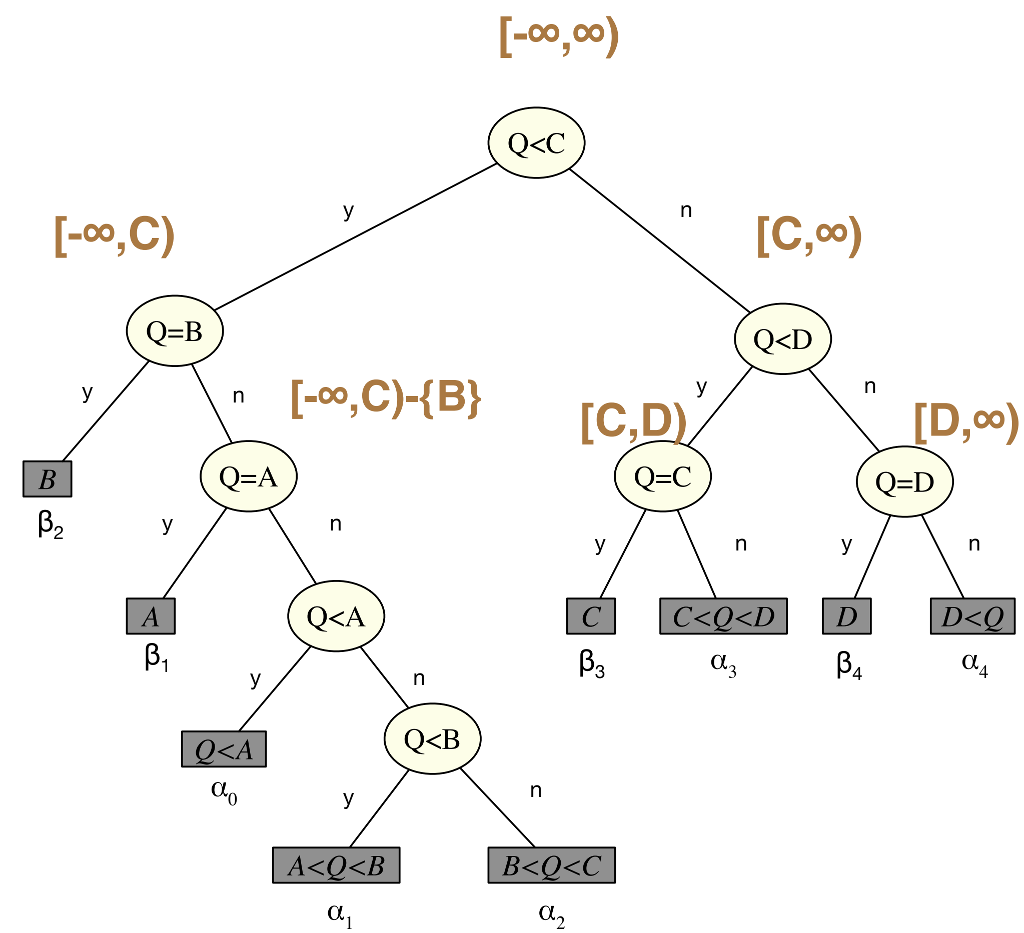

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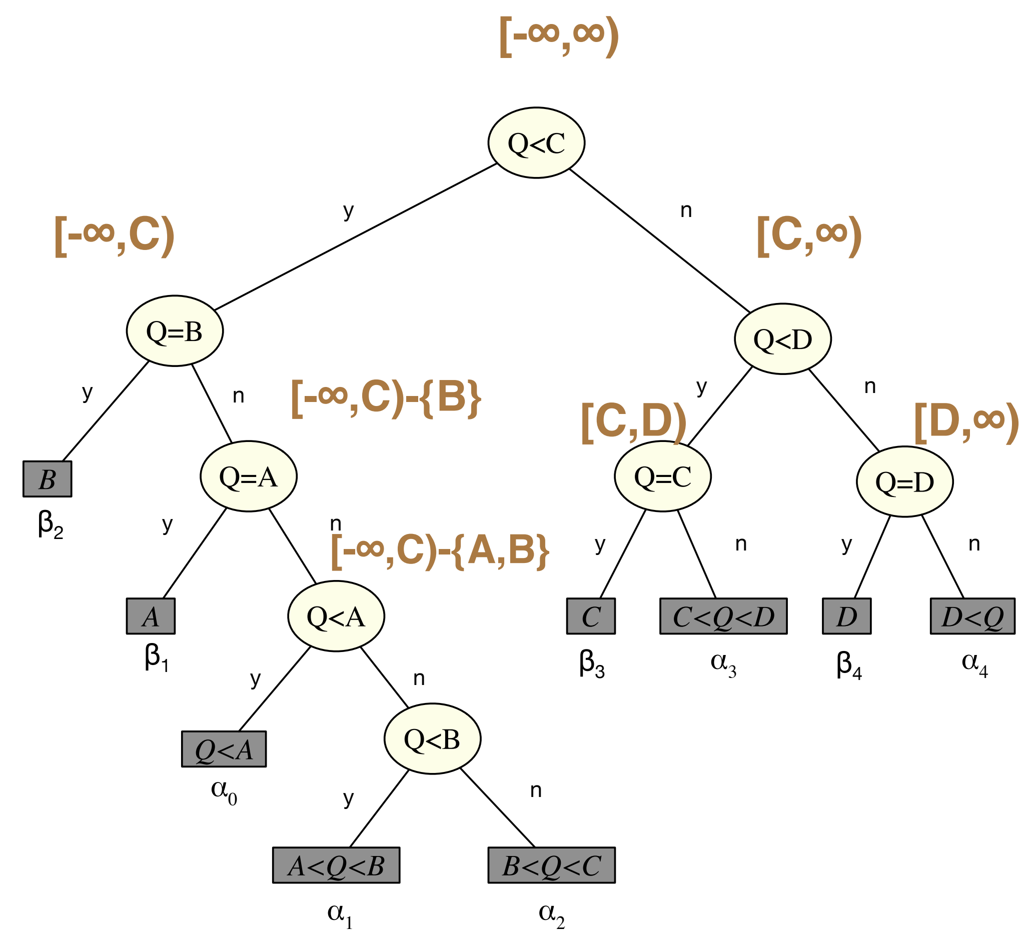

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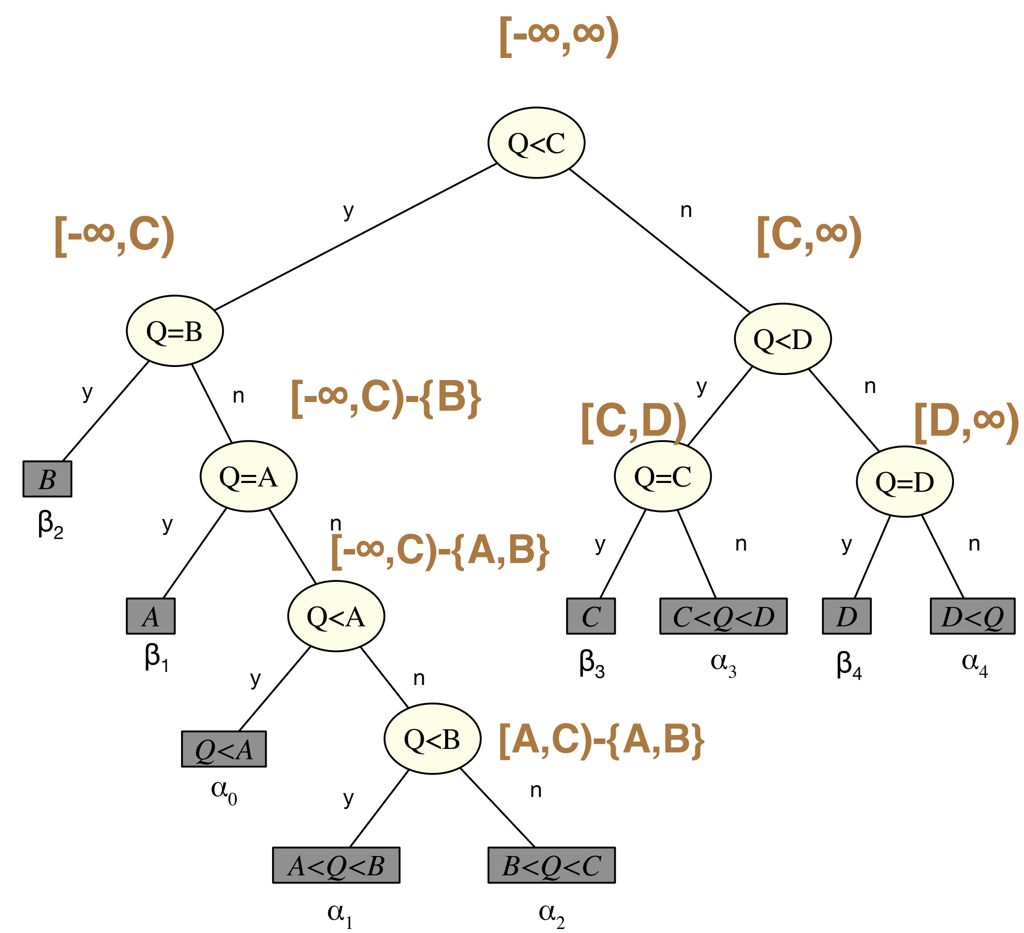

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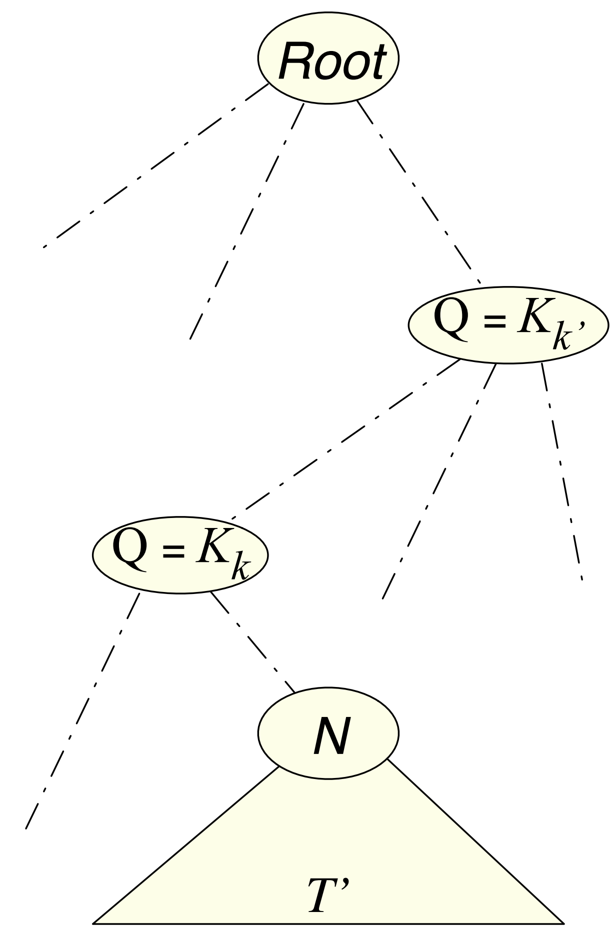

Structural Properties of OBCSTs

- Range associated with Node N is $[K_i, K_j)$ with some (h) keys K_k removed.
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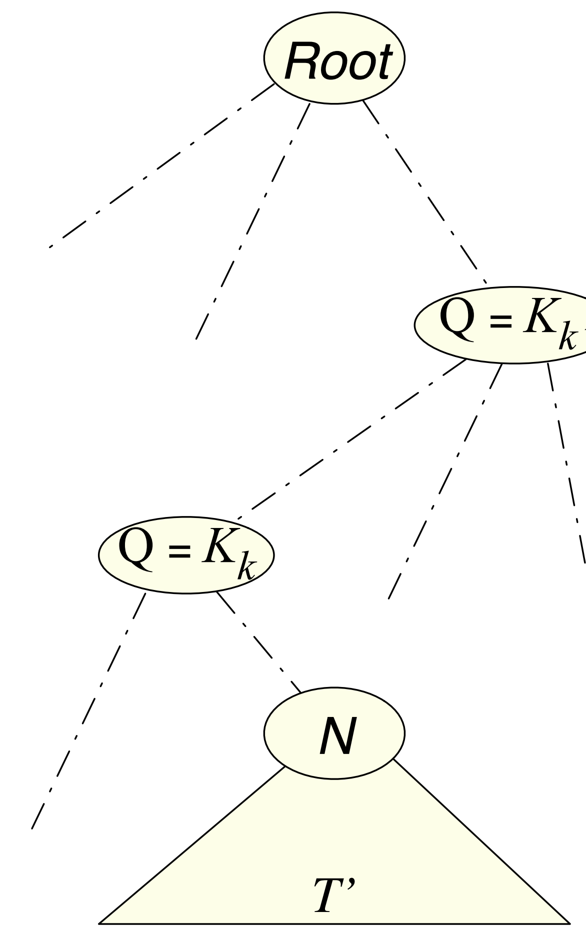

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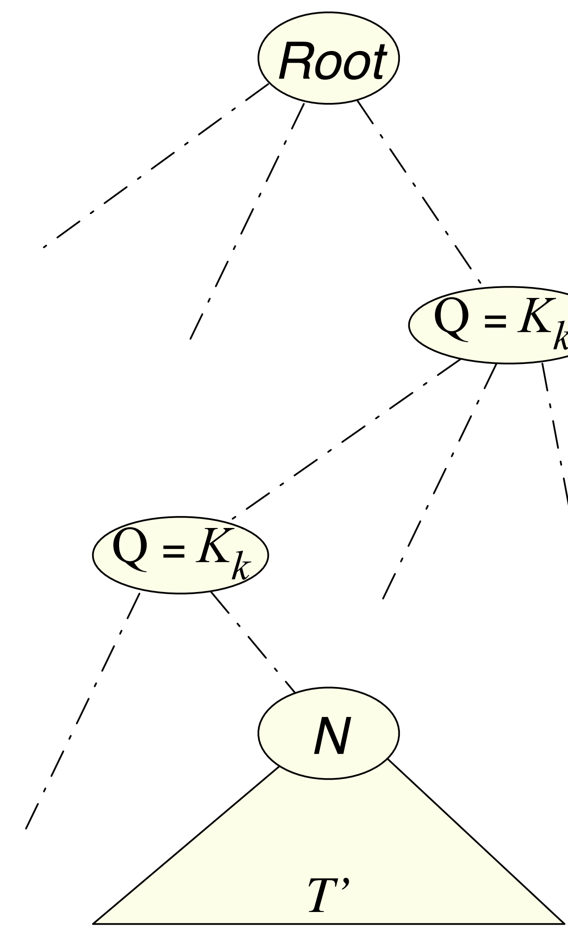
- K_k removed are s.t. $(Q=K_k)$ are equality nodes on path from N to root (that fall within $[K_i,K_j)$)
- From previous Lemma, if T is an OBCST, β_i of nodes path to N are larger than β_i of all equality nodes in T'.
- $\forall k$, $(Q=K_k)$ appears somewhere in T.

  Immediately implies that the h missing keys must be the largest weighted keys in $[K_i,K_j)$



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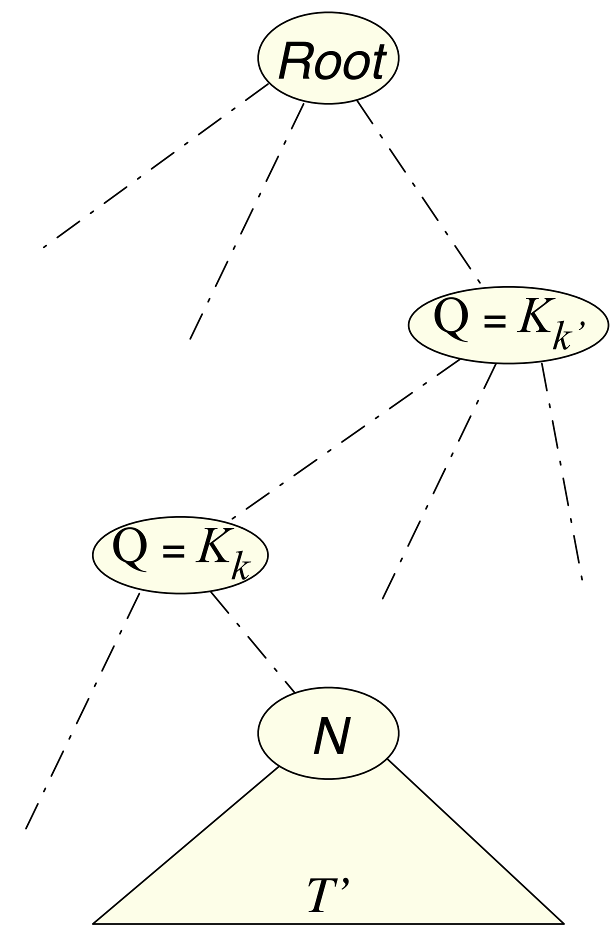
- Define punctured range **[i,j: h)** to be *range* $[K_i,K_j)$ *with the h highest weighted keys in* $[K_i,K_j)$ *removed*



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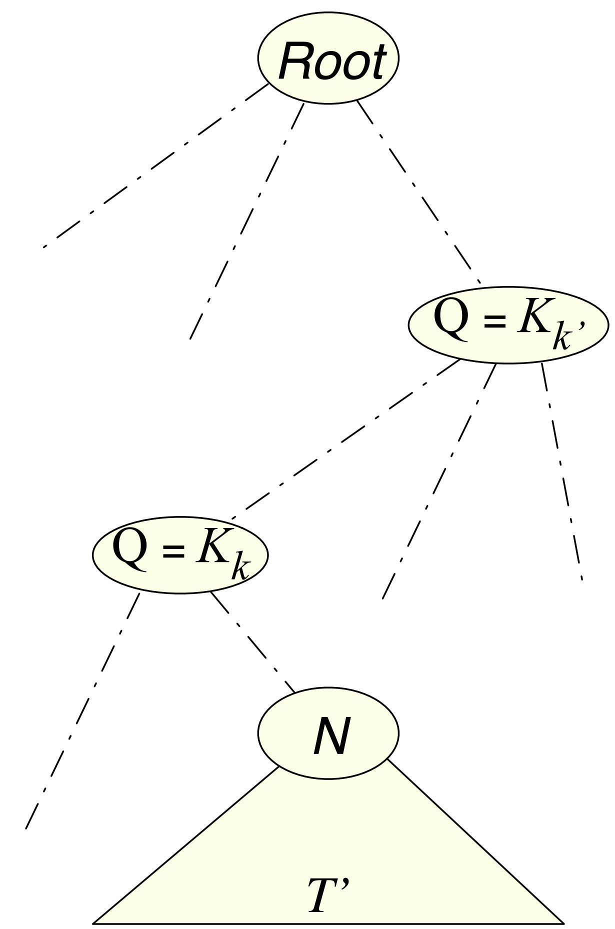
- Define punctured range **[i,j: h)** to be *range $[K_i,K_j)$ with the h highest weighted keys in $[K_i,K_j)$ removed*
- **=> every range associated with an internal node of an OBCST is a punctured range**



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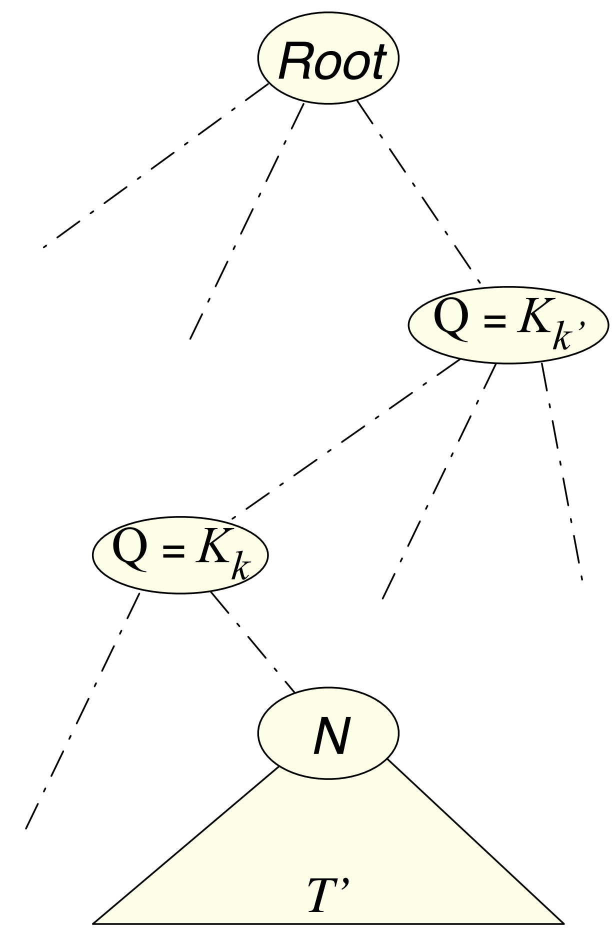

Structural Properties of OBCSTs

- **[i,j: h)** is *range $[K_i,K_j)$ with the h highest weighted keys in $[K_i,K_j)$ removed*

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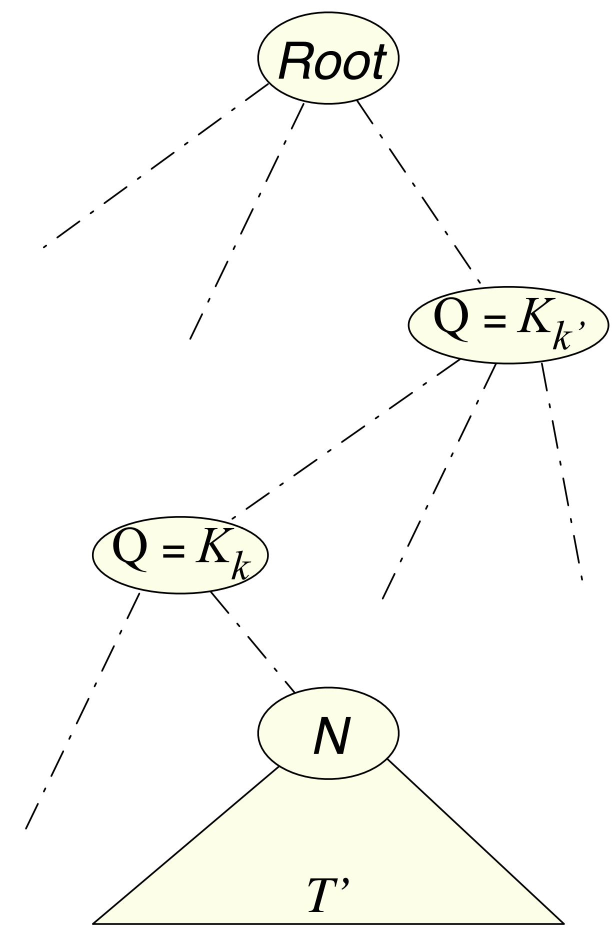


Structural Properties of OBCSTs

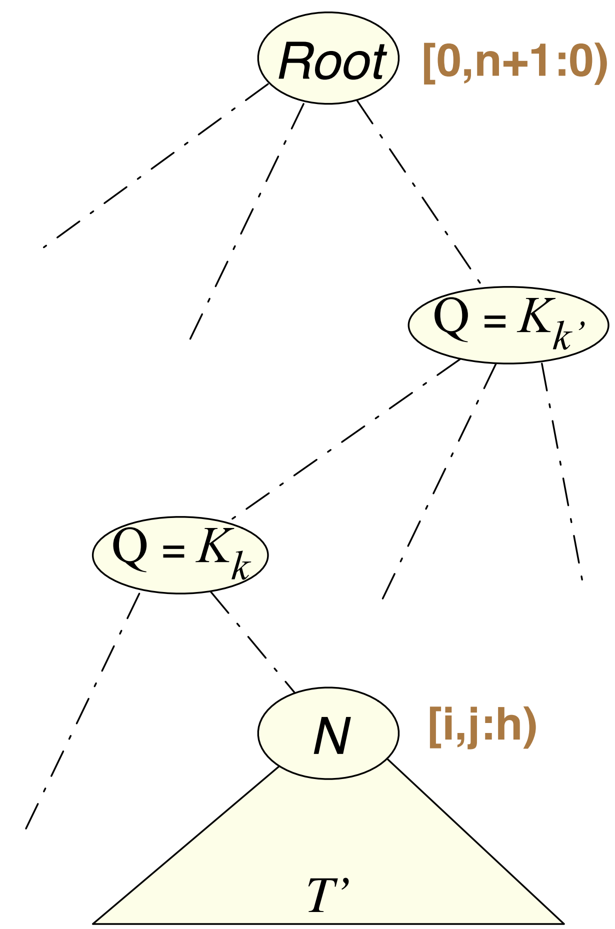
- **[i,j: h)** is *range* [K_i,K_j) *with the h highest weighted keys in* [K_i,K_j) *removed*

- *Range associated with an internal node of an OBCST is some* **[i,j: h)**

- Define **OPT(i,j: h)** to be the *cost of an optimal BCST for range [i,j: h)*

- *Goal* is to find **OPT(0,n+1: 0)** and associated tree

- Will use Dynamic programming to fill in table. Table has size $O(n^3)$ We will (recursively) evaluate **OPT(i,j: h)** in O(j-i) time, yielding a $O(n^4)$ algorithm.



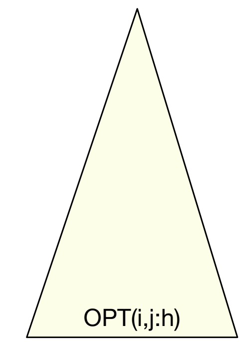
Outline

- History
  - Binary Search Trees
  - Hu-Tucker Trees
  - AKKL Trees
- Optimal Binary Comparison Search Trees with Failures
  - Problem Models
  - List of New Results
- New Results
  - The Main Lemma
  - Structural Properties of OBCSTs
  - Dynamic Programming for OBCSTs
  - Proof of The Main Lemma (Sketch)
- Extensions and Open Problems

Dynamic programming for OBCSTs

Dynamic programming for OBCSTs

- Let T be an OBCST for [i,j: h)
- T Has two possible structures



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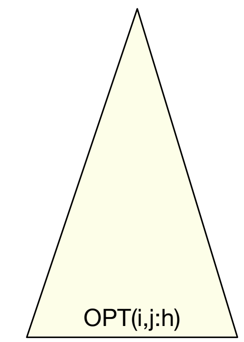
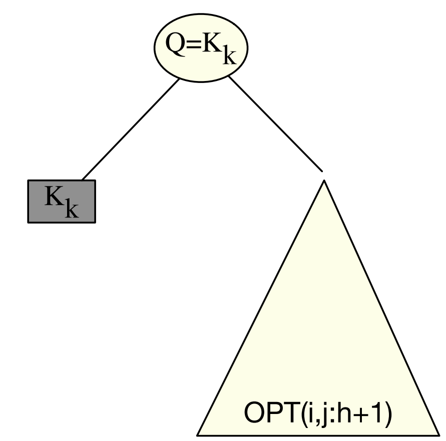


1. Root is a $(Q=K_k)$



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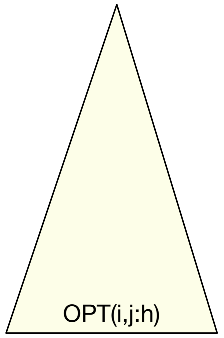

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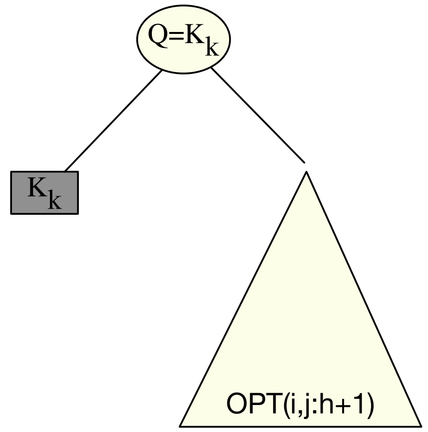

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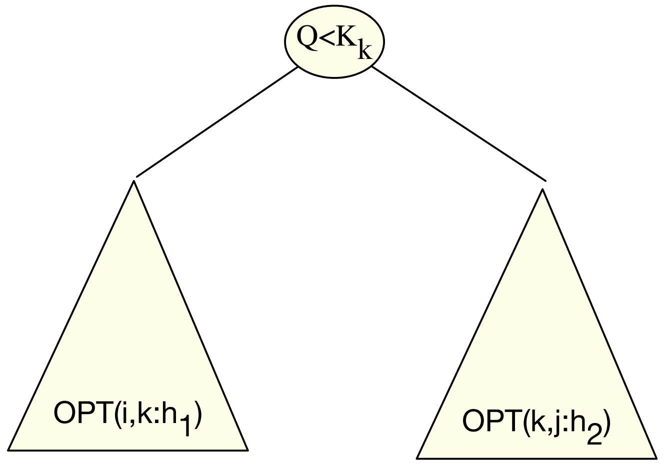

Dynamic programing for OBCSTs

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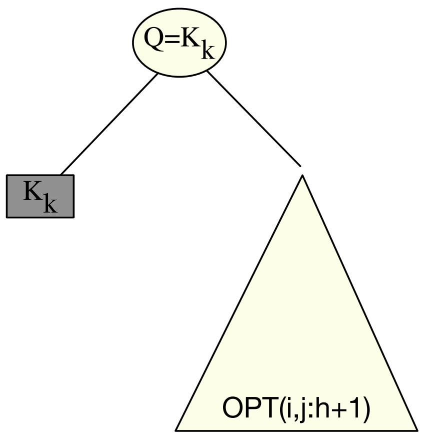

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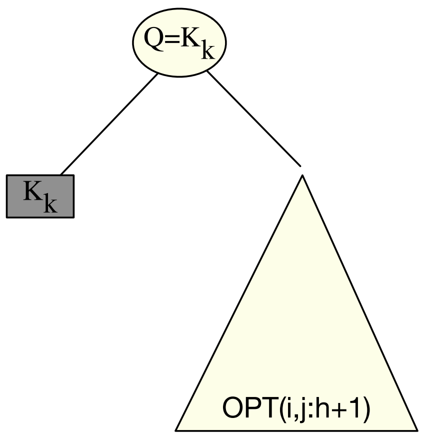

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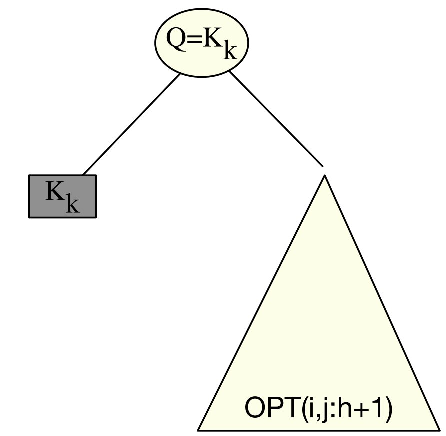

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Cost of full tree is sum of

- cost of left subtree *0*
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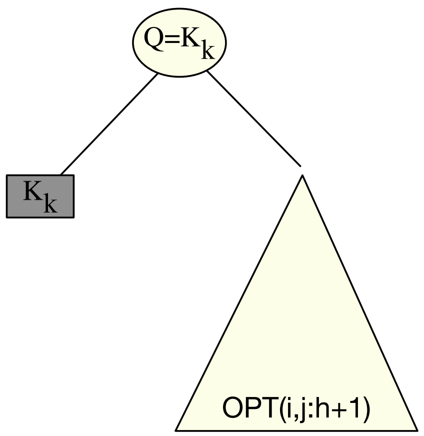

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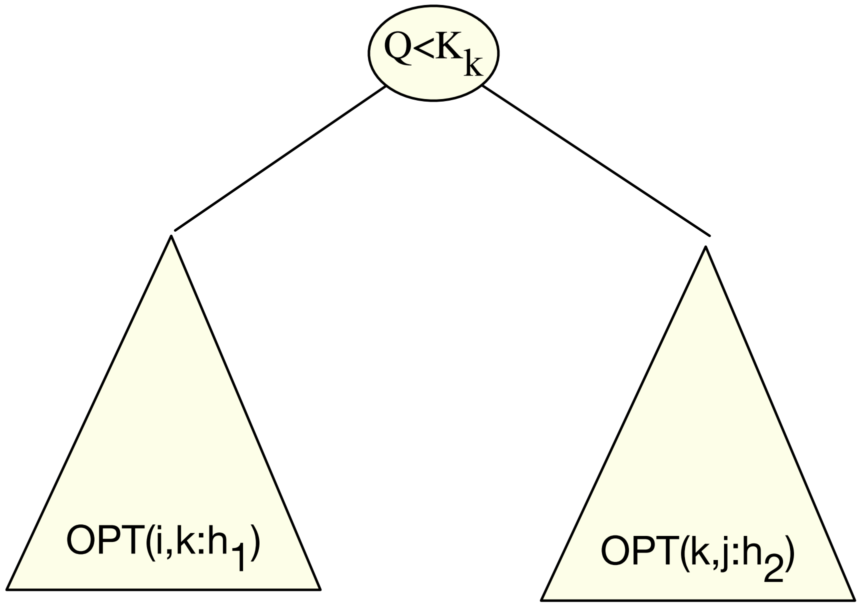
Dynamic programing for OBCSTs

2. Root of OPT(i,j: h) is a $(Q<K_k)$



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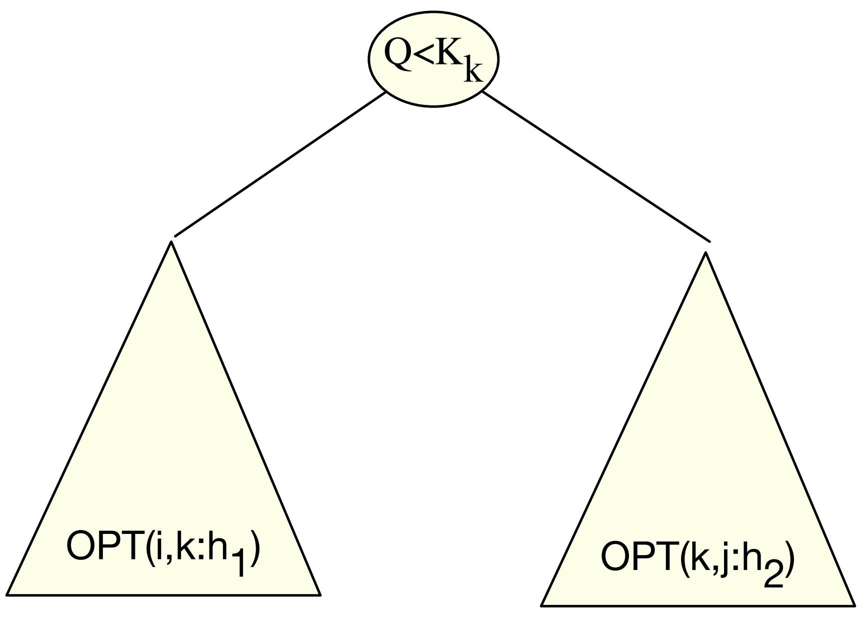

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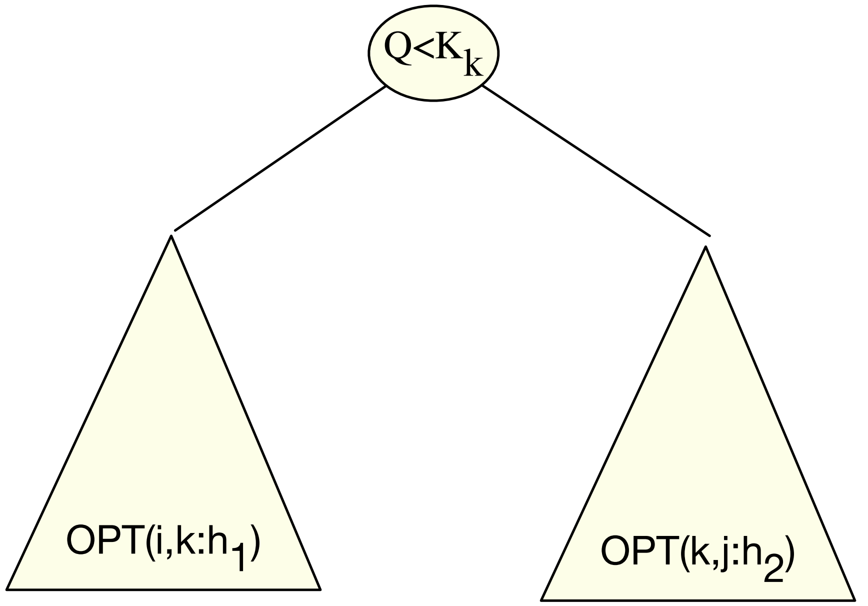

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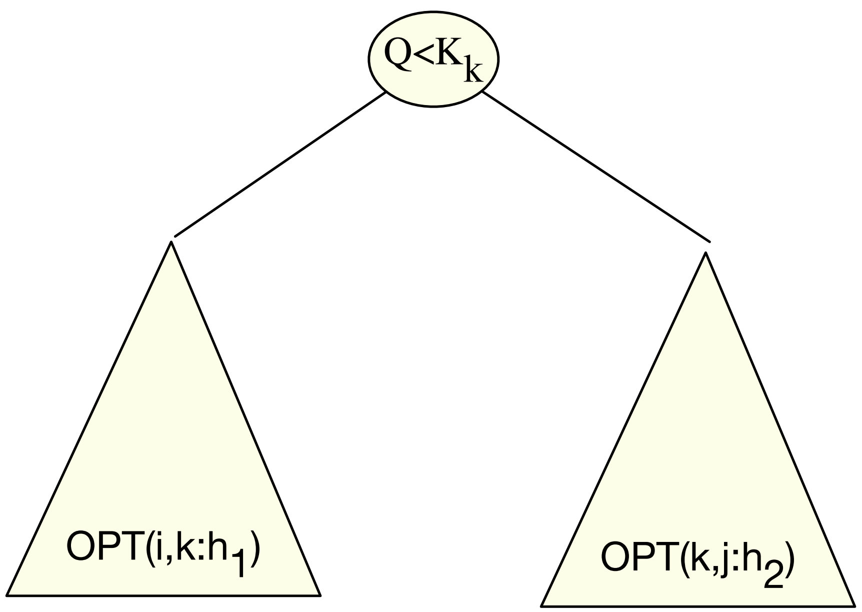

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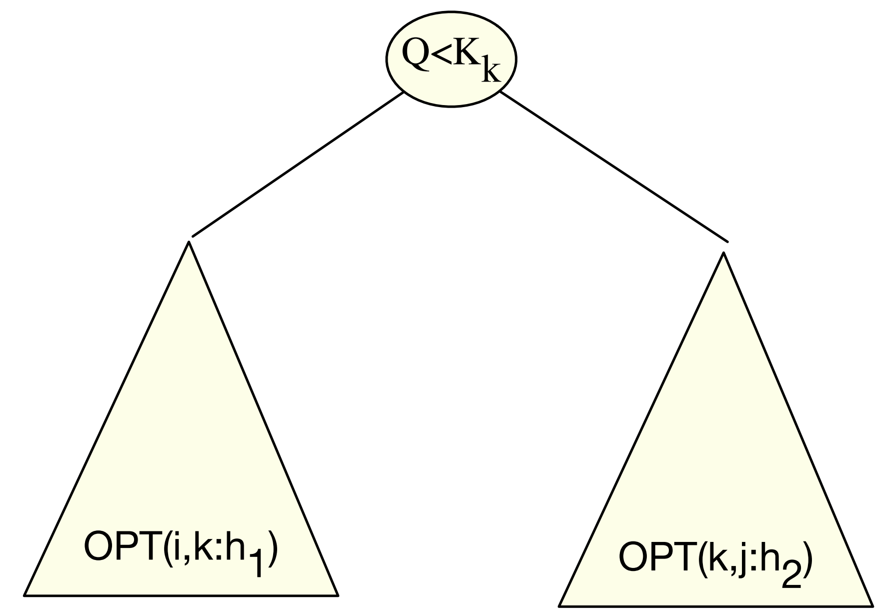

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Don't know what *k* is, so minimize over all possible k

$$SPLIT(i,j:h) = \min_{i<k<j} \{W_{i,j:h} + OPT(i,k:h_1(k)) + OPT(k,j:h_2(k))\}$$

Dynamic programing for OBCSTs

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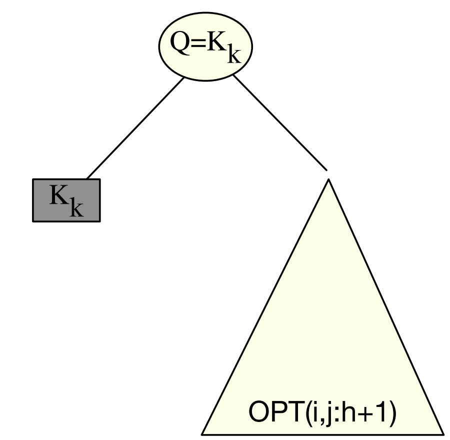

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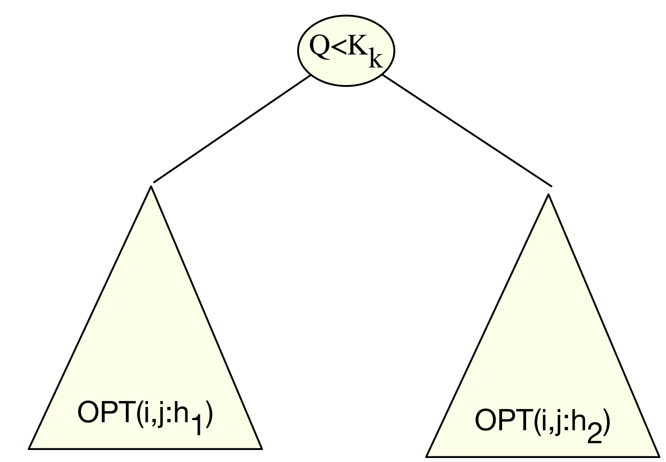

Dynamic programing for OBCSTs

OPT(i,j: h) has two possible structures

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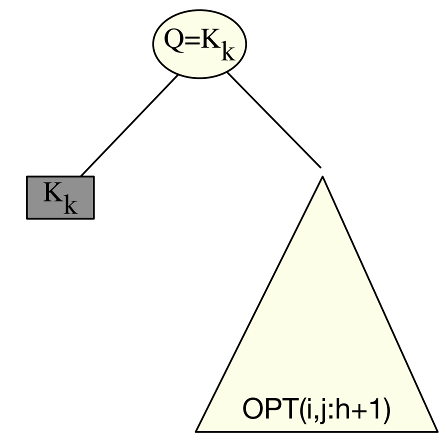

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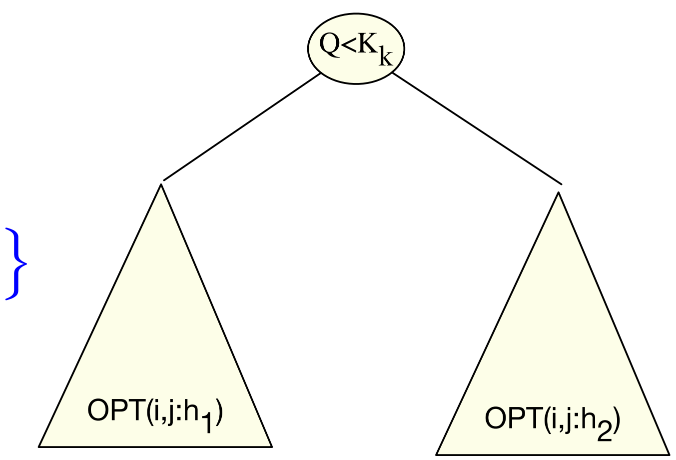

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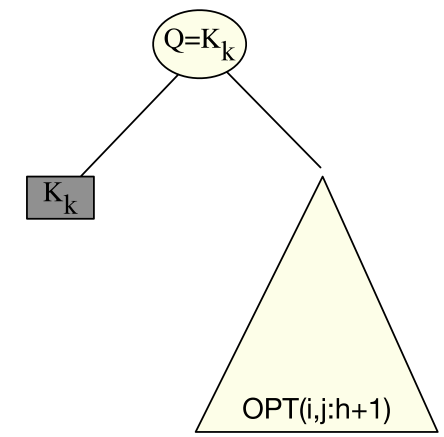

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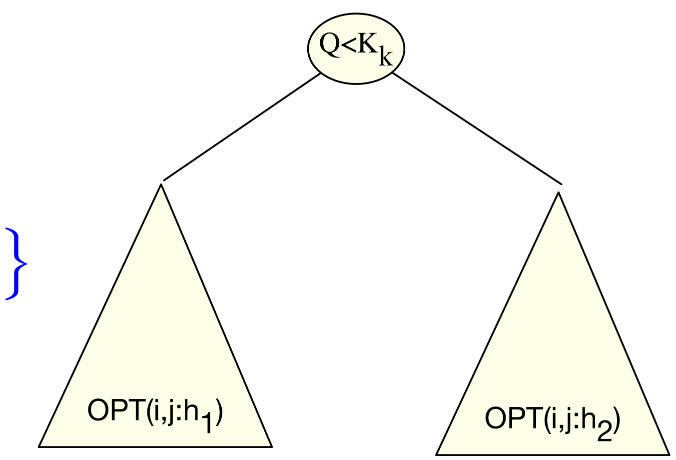

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This immediately implies

$$OPT(i,j:h) \geq \min\left(EQ(i,j:h),\, SPLIT(i,j:h)\right)$$

Dynamic programing for OBCSTs

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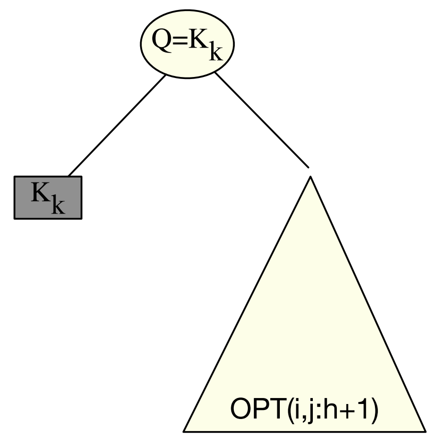

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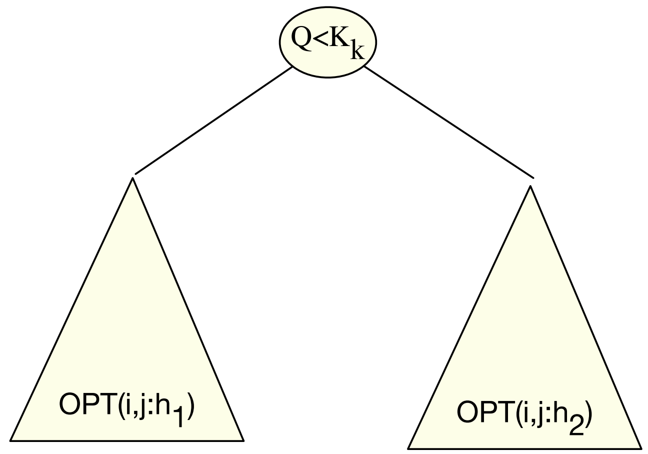

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This immediately implies

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But every case seen can construct a BCST with that cost, so

$$OPT(i,j:h) = \min\left(EQ(i,j:h),\ SPLIT(i,j:h)\right)$$

Dynamic programing for OBCSTs

$$OPT(i, j : h) = \min\left(EQ(i, j : h),\, SPLIT(i, j : h)\right)$$

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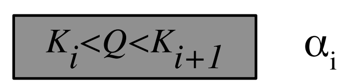
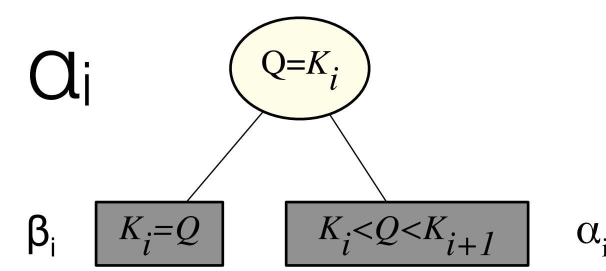


OPT(i,i+1,0)= $\beta_i + \alpha_i$



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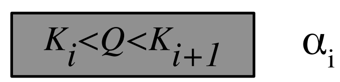

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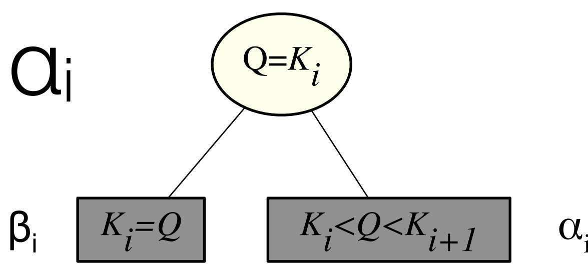

Comments

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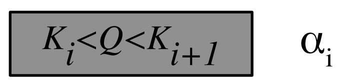

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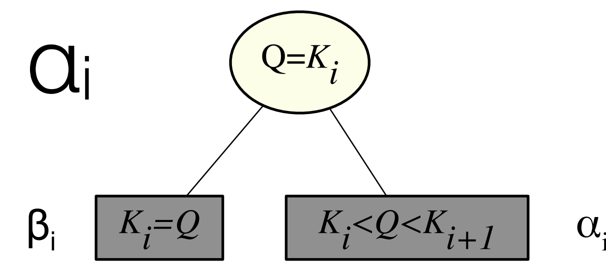

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- Must restrict h ≤ j-i (can't have more holes than keys in interval)

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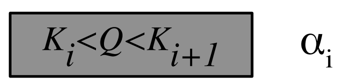

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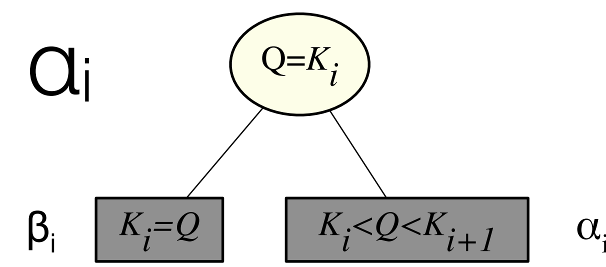

Comments

- Must restrict h ≤ j-i (can't have more holes than keys in interval)
- Need to fill in table in proper order, e.g.,
  (a) d= 0 to n, (b) i=0 to n-d, j=i+d+1, (c) h =(j-i) downto 0

Dynamic programing for OBCSTs

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Set initial conditions for ranges OPT(i,i+1,*)

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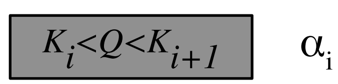

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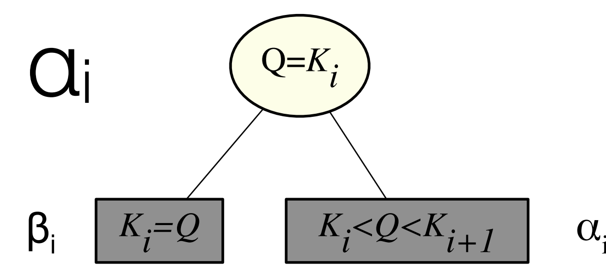

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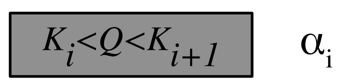

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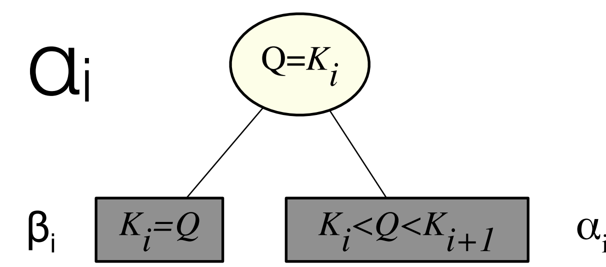

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- OPT(0,n+1:0) is optimal cost. Use standard DP backtracking to construct corresponding optimal tree

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- Both (A) and (C) can be implemented in O(1) time without knowing ϵ
- Perturbed algorithm has same asymptotic running time as regular one

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- If C={<, ≤}, ranges have no holes and problem can be solved in O(n log n) similar to Hu-Tucker

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  - Binary Search Trees
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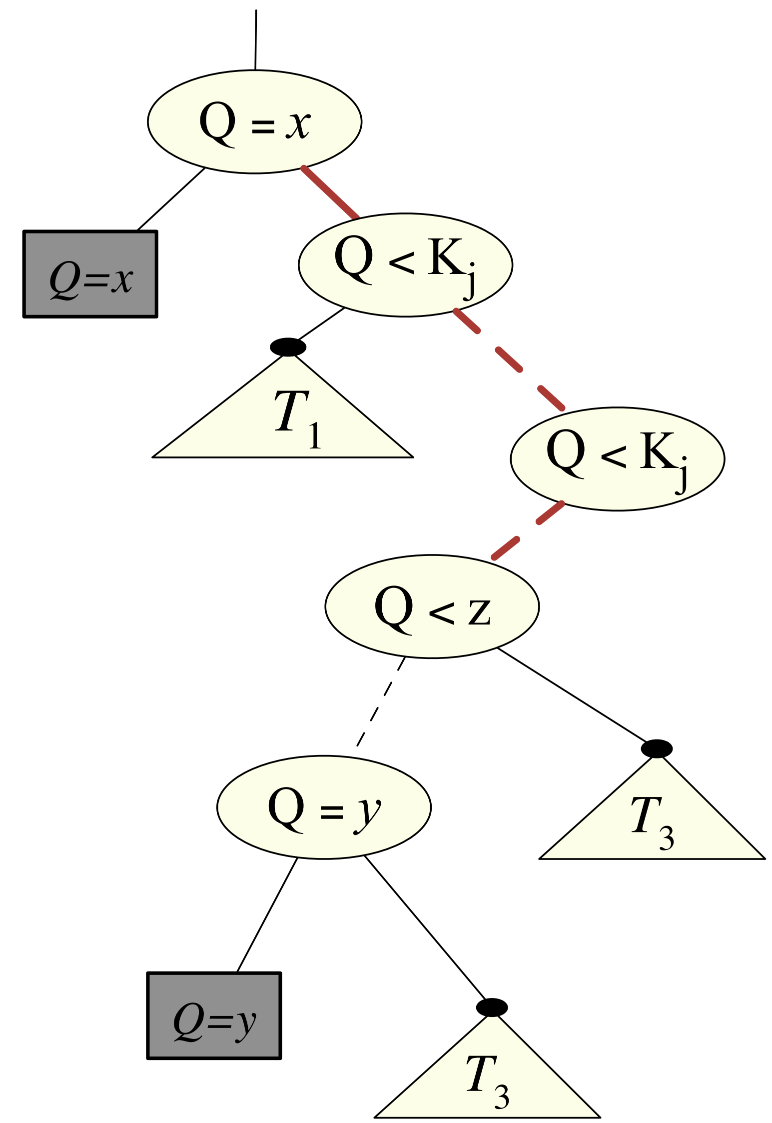
  - The Main Lemma
  - Structural Properties of OBCSTs
  - Dynamic Programming for OBCSTs
  - Proof of The Main Lemma (Sketch)
- Extensions and Open Problems

Proof of Main Lemma



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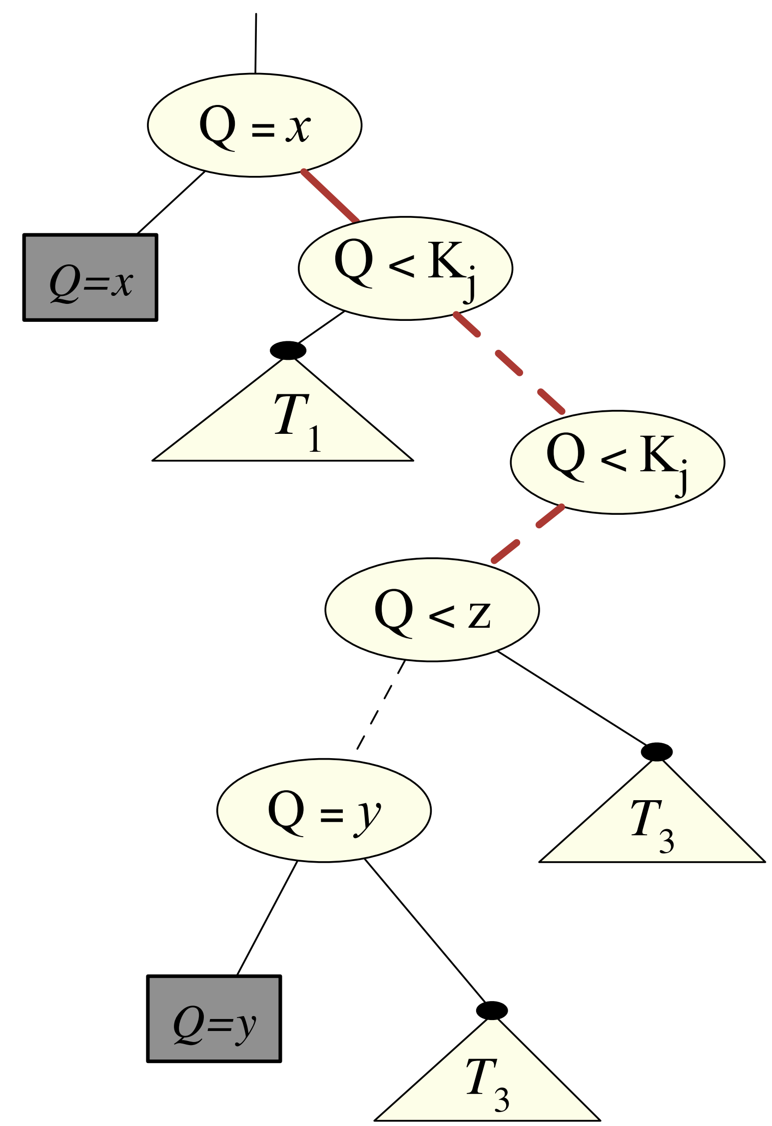

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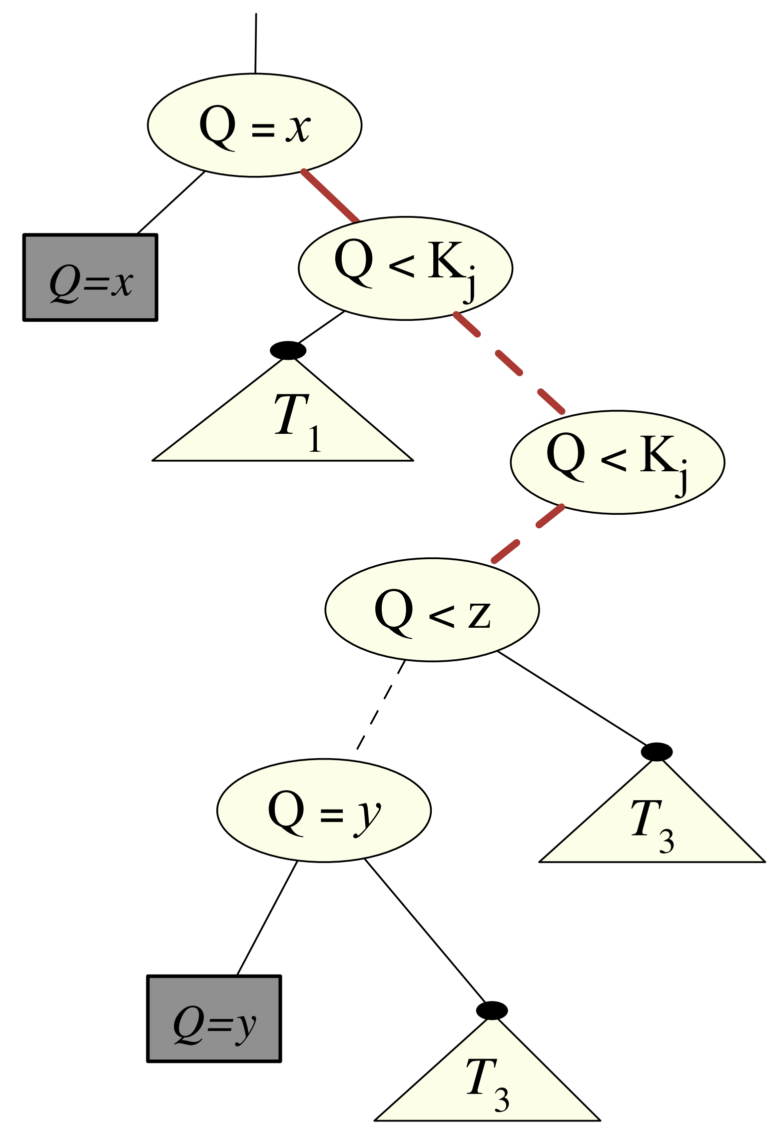

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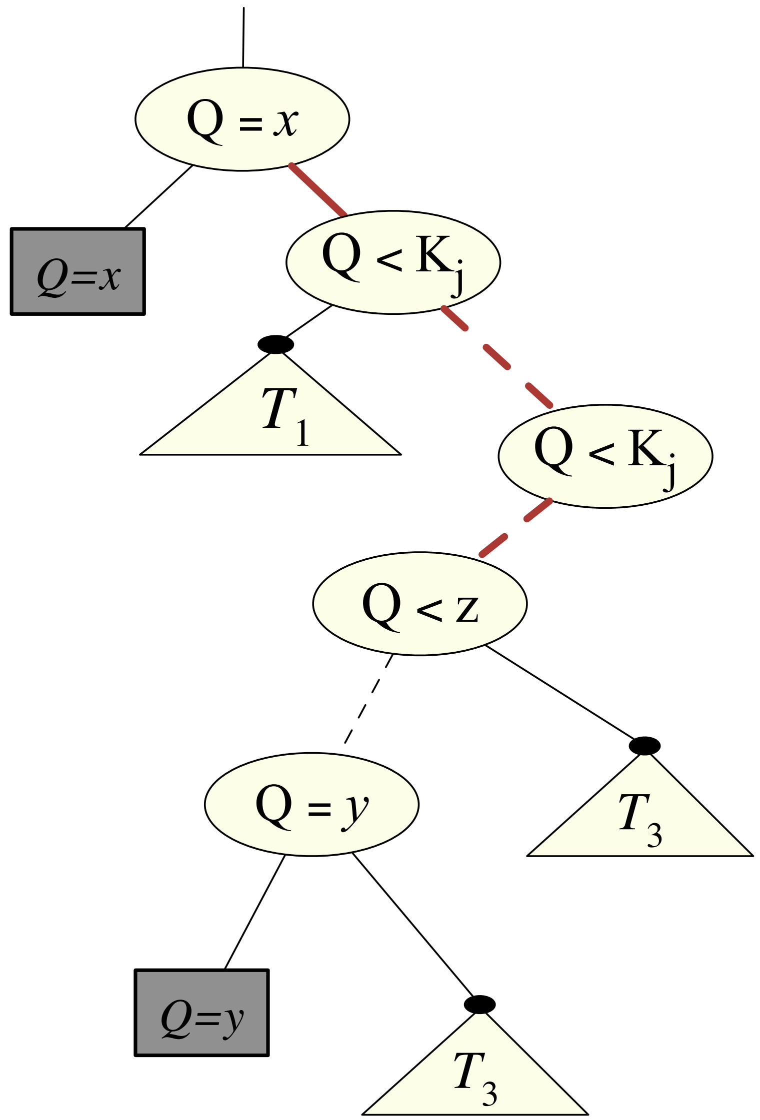

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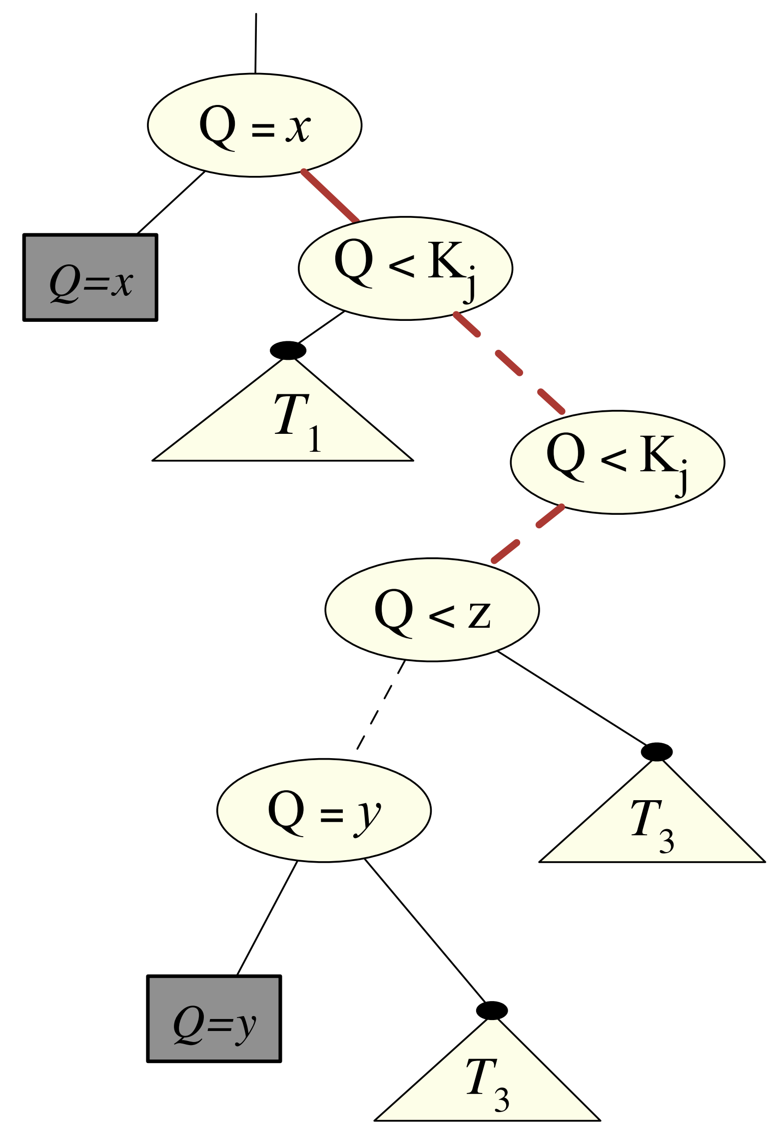

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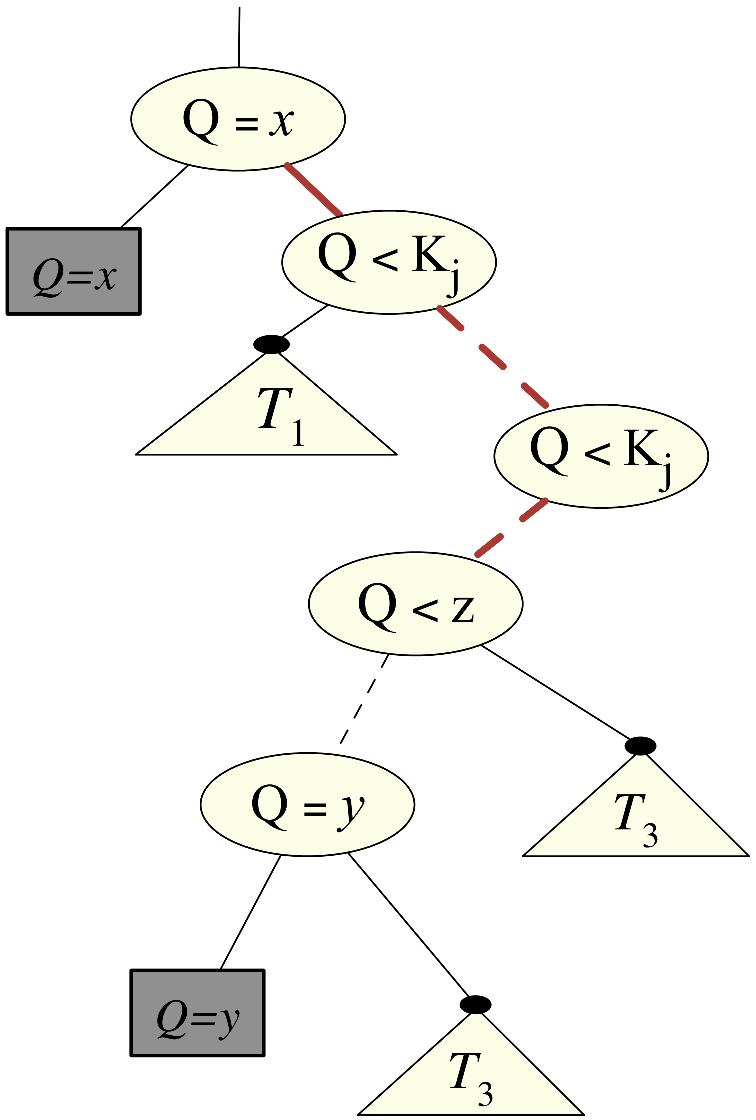

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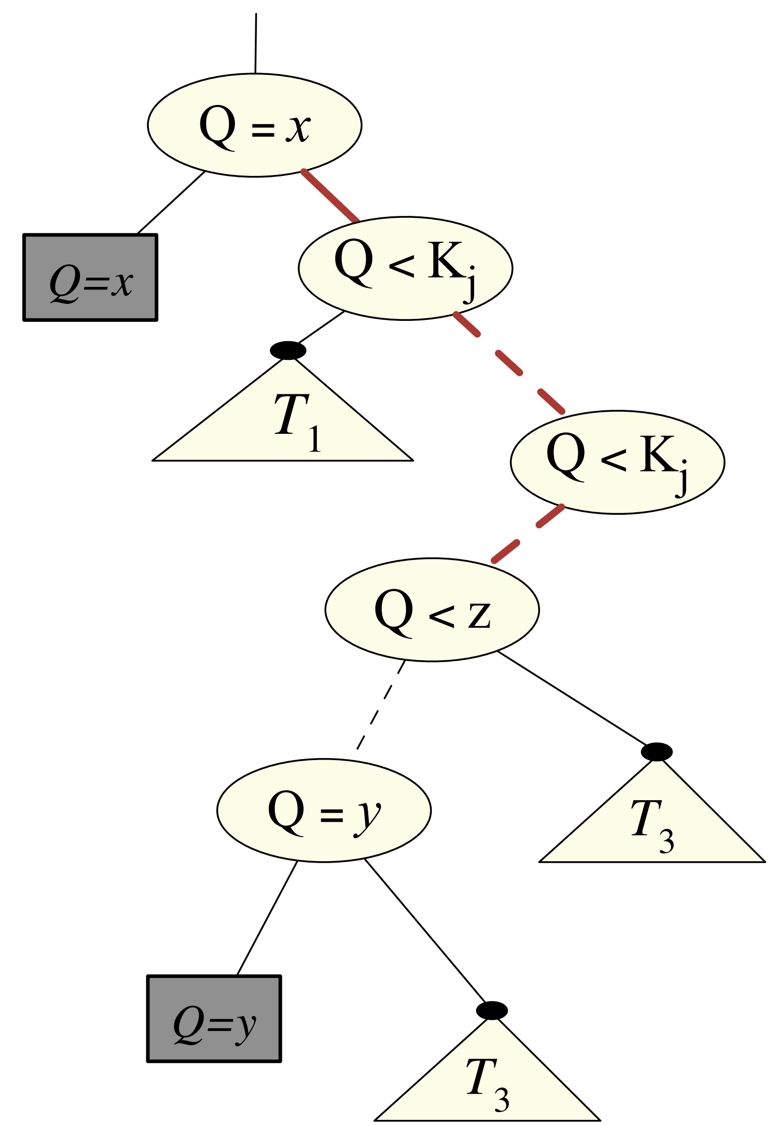

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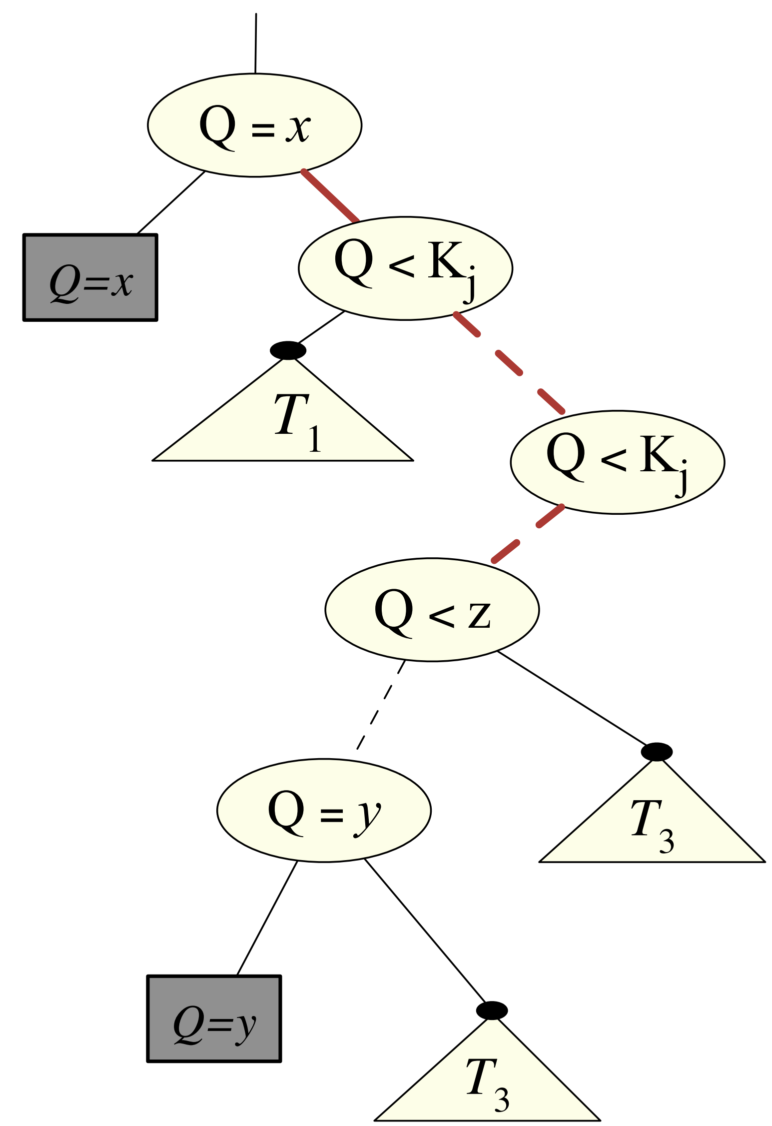
- x,y ∈ Range((Q=x)) by definition
  If x,y ∈ Range((Q=y))
  then could swap (Q=X) and (Q=y)
  to get cheaper tree.



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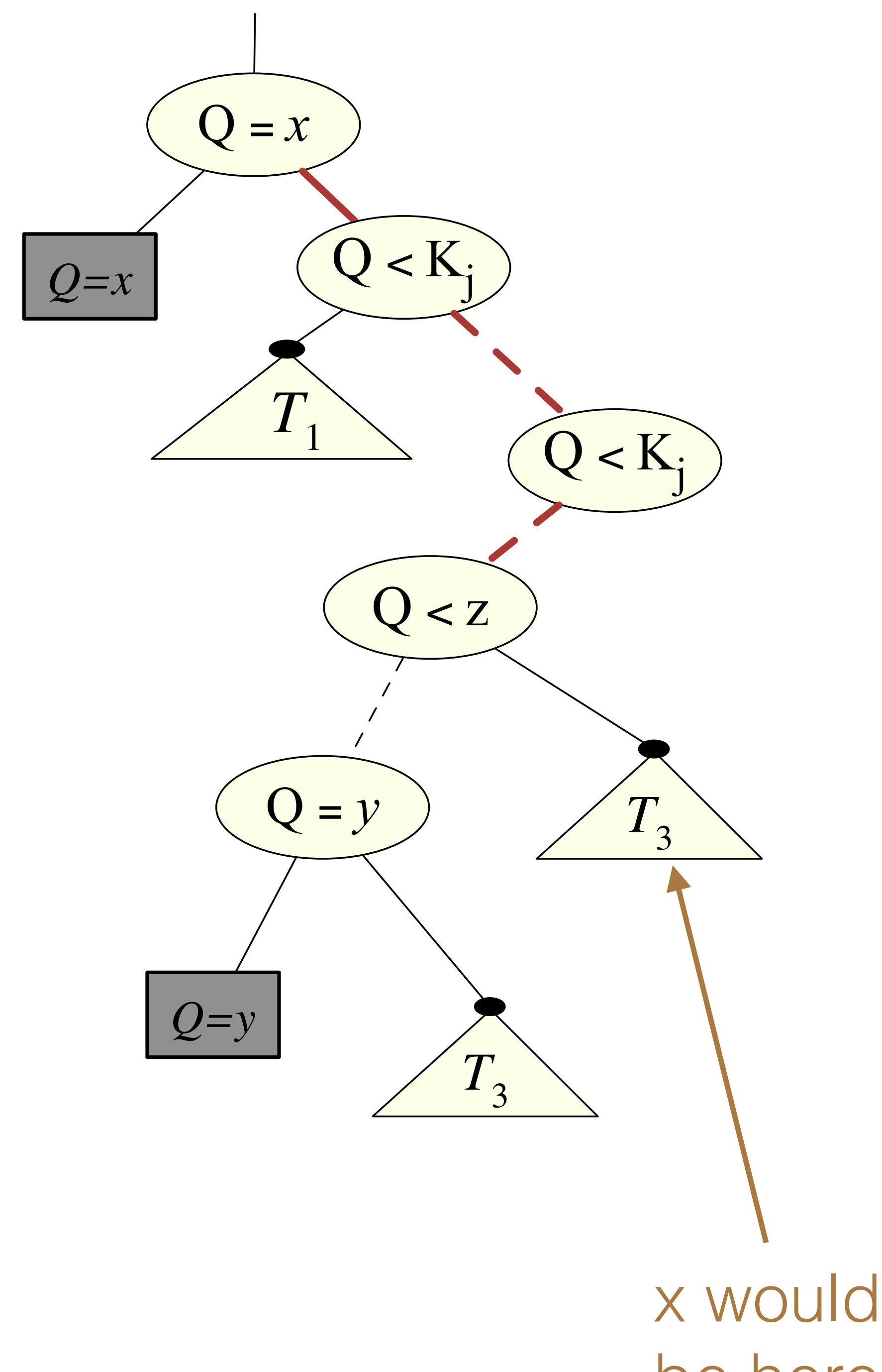

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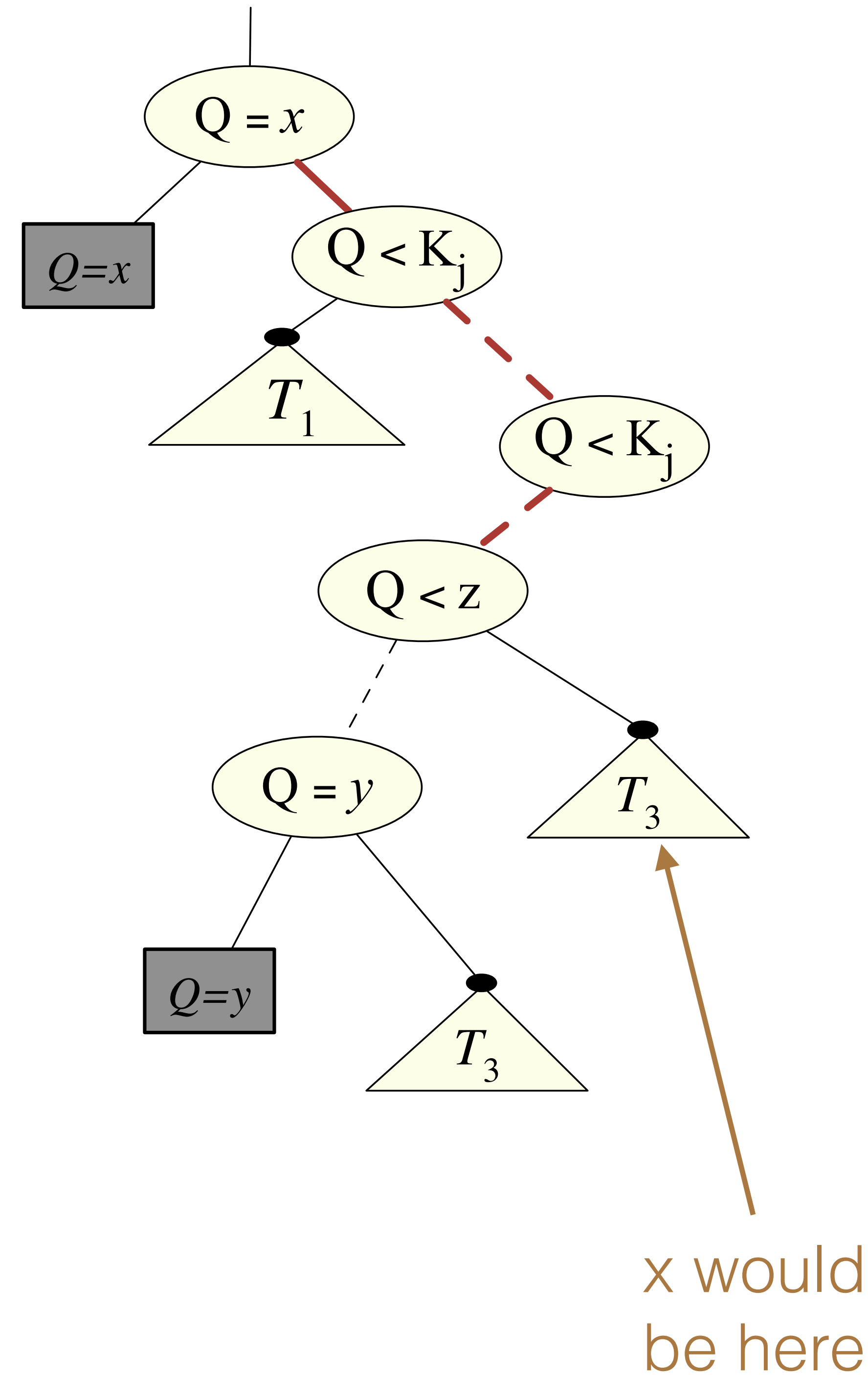

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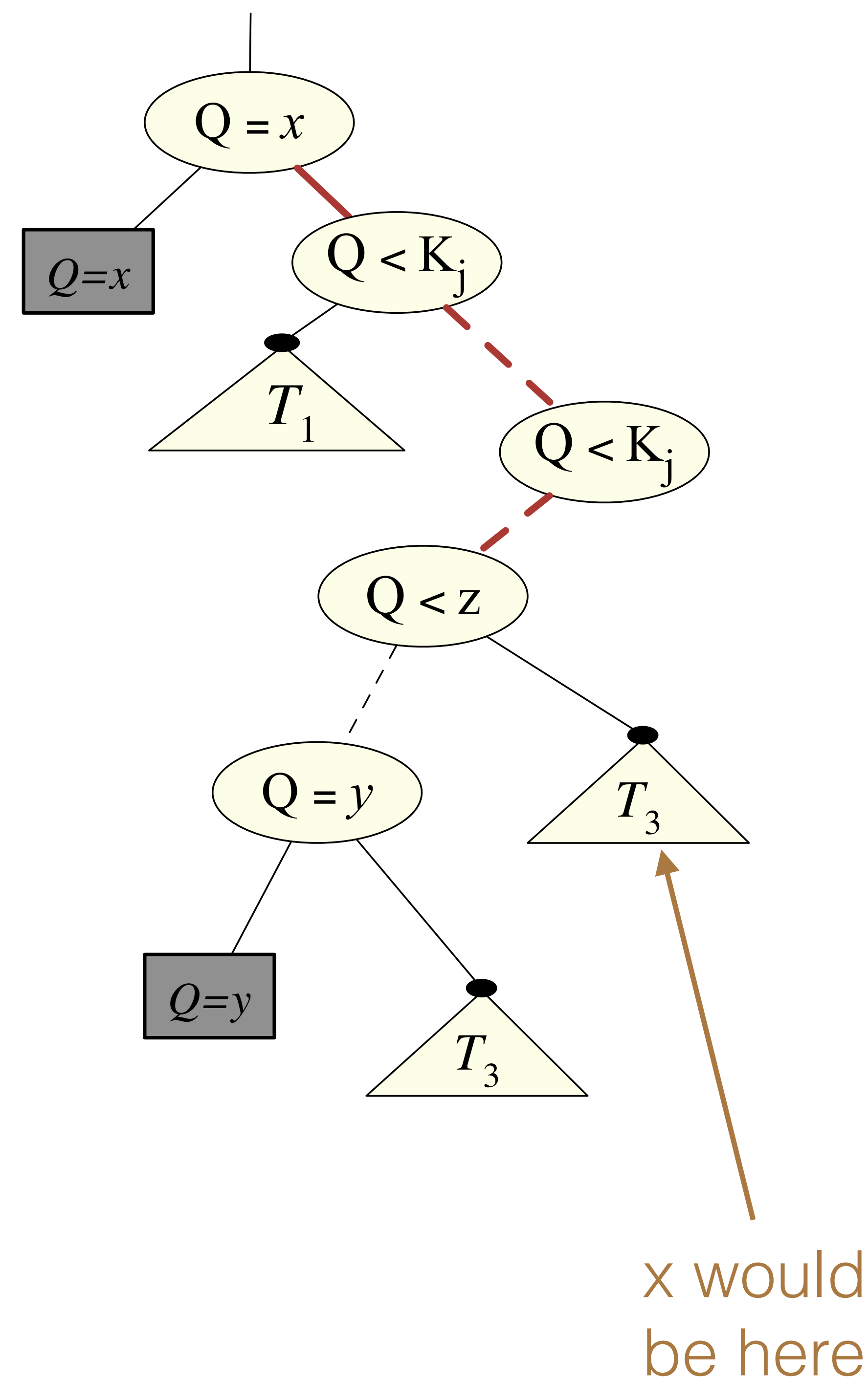
- Since x∉ Range((Q=y)) => Path (Q=x) to (Q=y) contains (Q<z) s.t z's children's ranges are [i,z,h'), [z,j,h'') where y∈ [i,z) and x ∈[z,j). z is called *splitter*.

- P' is (red) path from (Q=x) to (Q=y)



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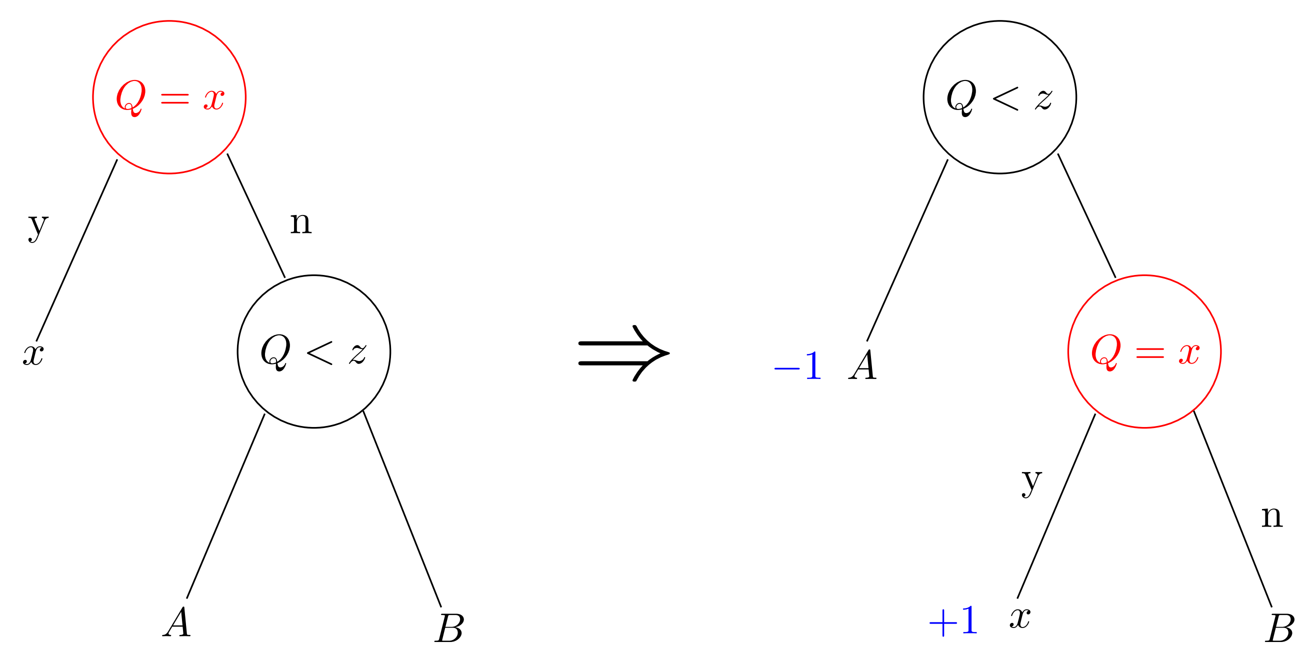

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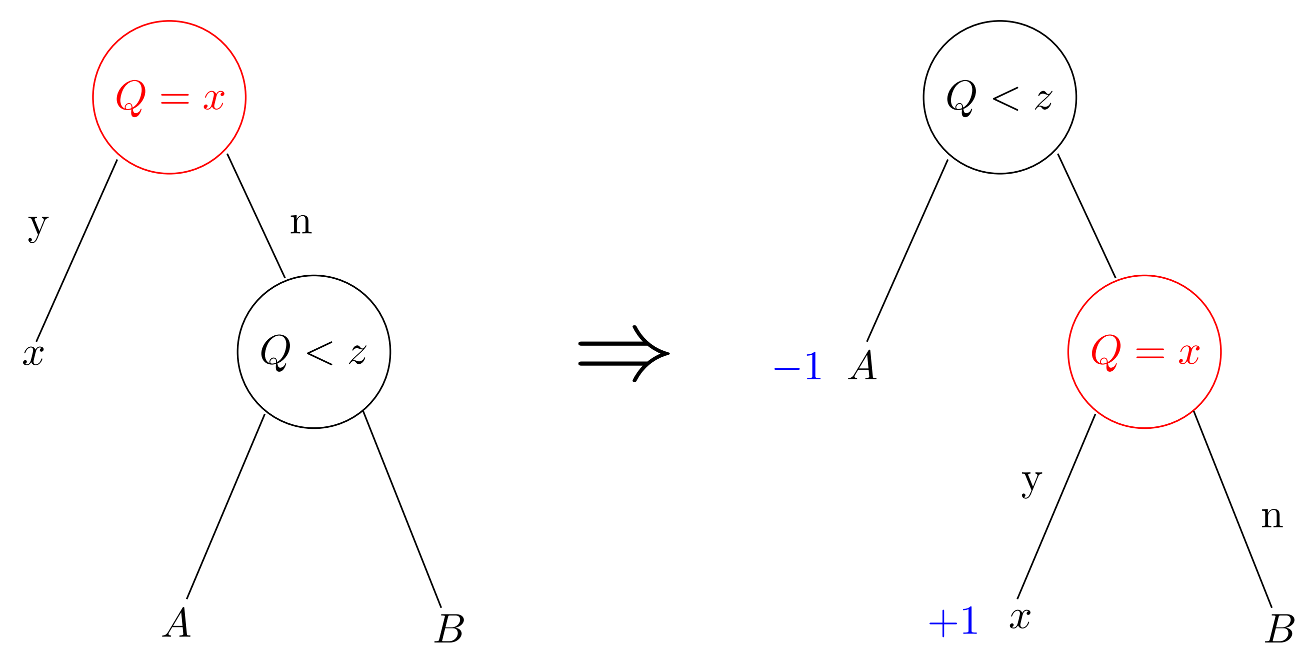
y∈A => Weight(A) ≥ $\beta_y > \beta_x$



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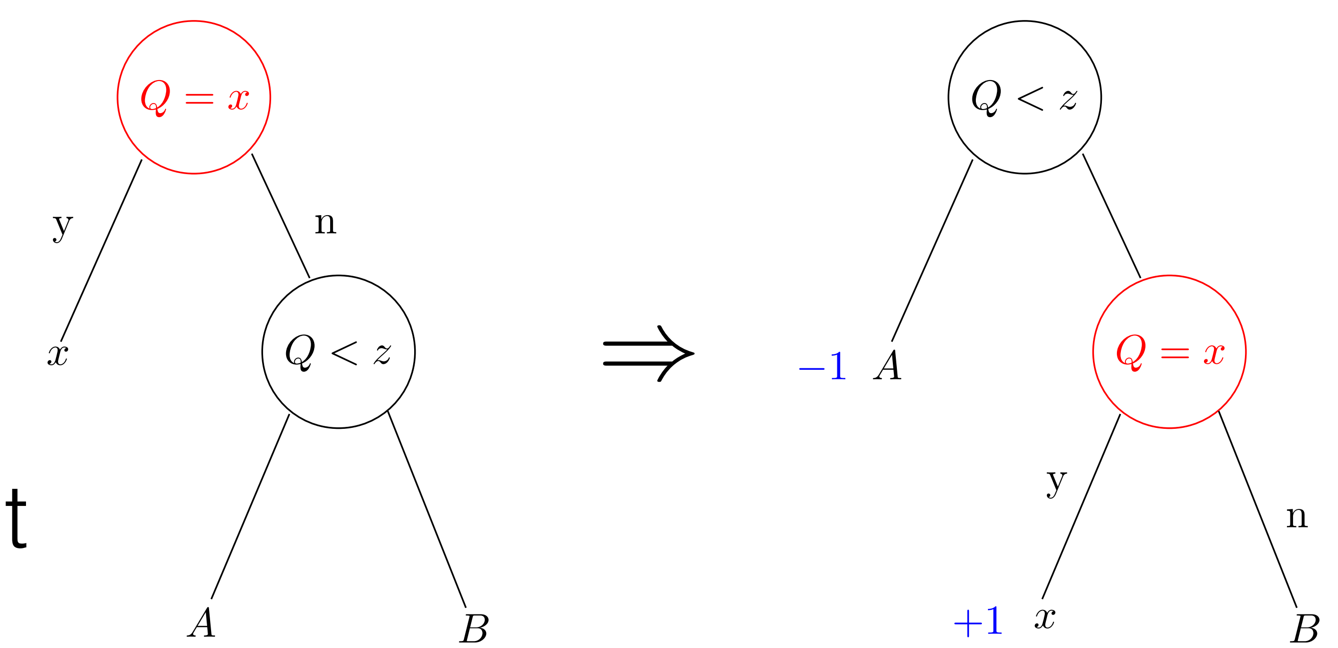

=> replacing left subtree by right subtree in T yields new BCST T' with lower cost than T, contradicting T being OBCST

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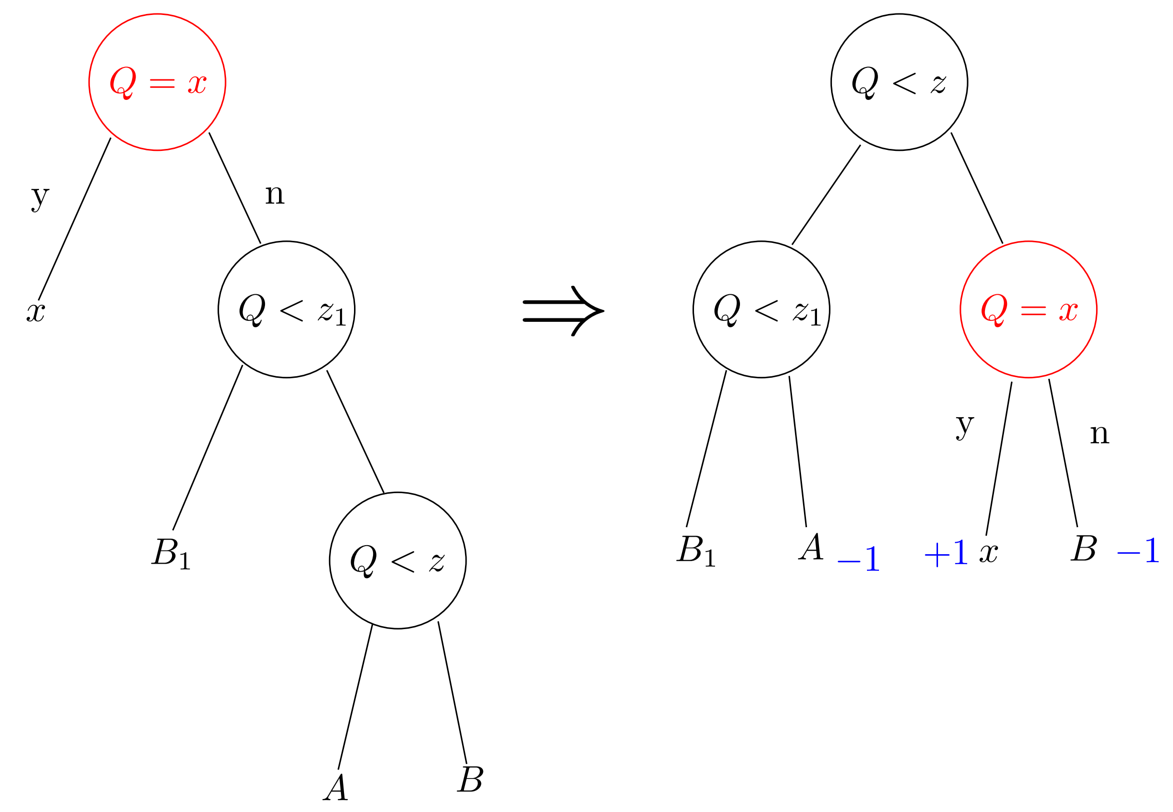

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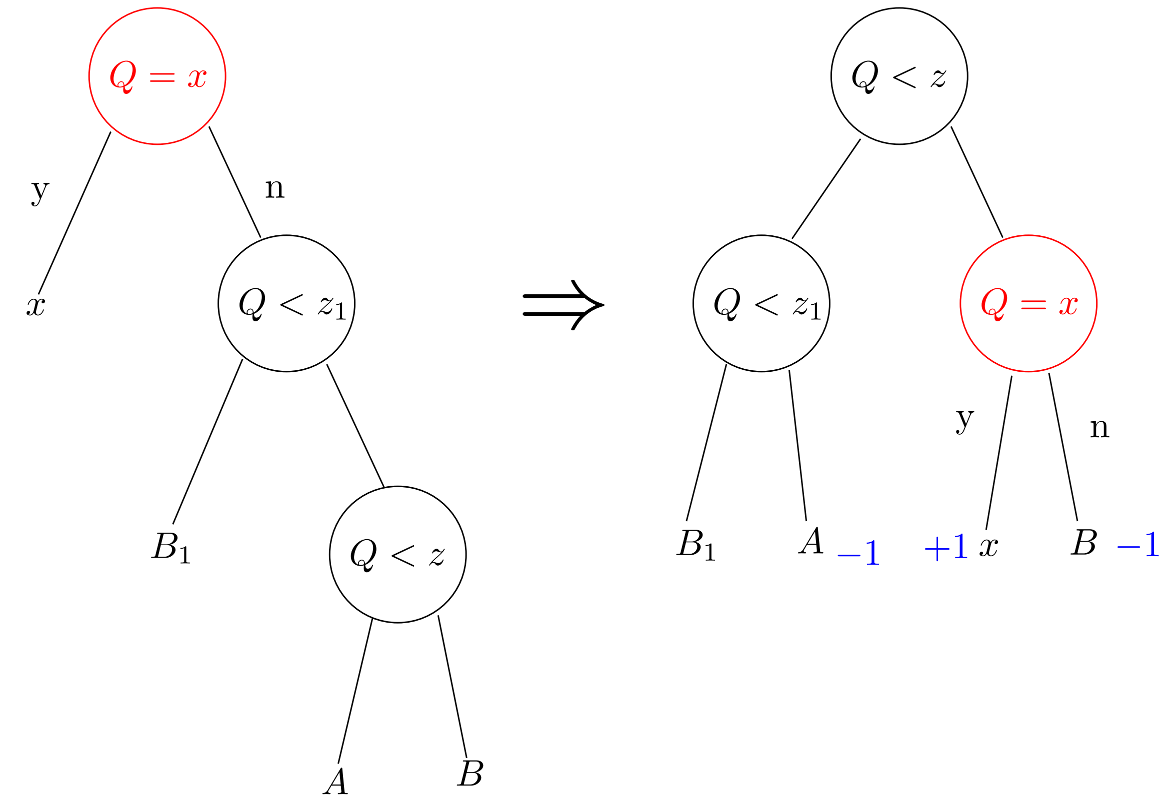

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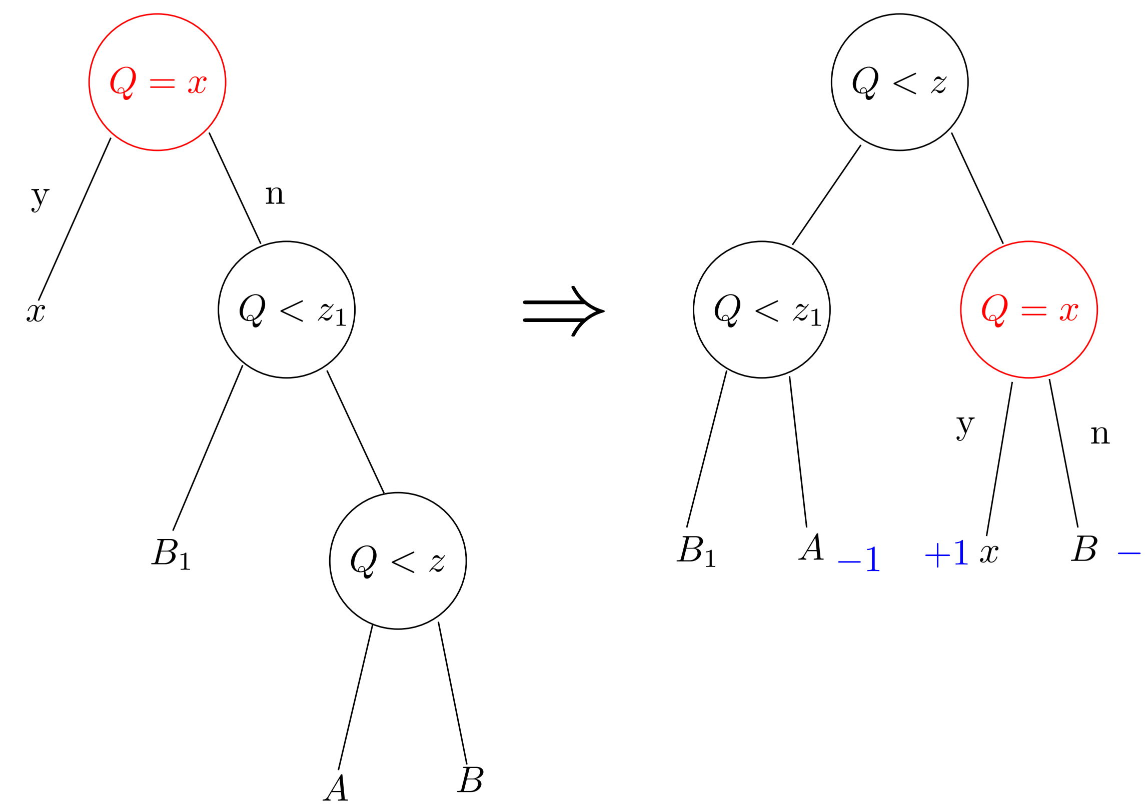

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    - Showed for each that assumptions allow replacing subtree rooted at (Q=x) with cheaper subtree for some range. Replacement leads to cheaper BCST, contradicting optimality of T

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- The full proof splits P' into 7 cases.
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Extensions & Open Problems

- If the β_i, α_i are probabilities (sum to 1) can show an O(n) algorithm that constructs BCST within **additive error 3 of optimal** for Exact Failure Case
  - Modification of similar algorithm for Hu-Tucker case.

- $O(n^4)$ is quite high for worst case.
  - Can we do better?