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Lopsided Trees, I: Analyses[1]

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Abstract. Lopsided trees are rooted, ordered trees in which the length of an edge from a node to its ith child depends upon the value of i. These trees model a variety of problems and have therefore been extensively studied. In this paper we combine analytic and combinatorial techniques to address three open problems on such trees:

- Given n, characterize the combinatorial structure of a lopsided tree with n leaves that has minimal external path length.
- Express the cost of the minimal external path length tree as a function of n.
- Calculate exactly how many nodes of depth $\le x$ exist in the infinite lopsided tree.

Lopsided trees model *Varn codes*, prefix free codes in which the letters of the encoding alphabet can have different lengths. The solutions to the first and second problems above solve corresponding open problems on Varn codes. The solution to the third problem can be used to model the performance of broadcasting algorithms in the postal model of communication. Finding these solutions requires generalizing the definition of Fibonacci numbers and then using Mellin-transform techniques.

Key Words. Varn codes, Fibonacci recurrences, Mellin transforms, Postal model.

1. Introduction. In this paper we discuss some properties of lopsided trees. A tree is said to be *lopsided* if it is a rooted, ordered (i.e., the children of each node are ordered) tree with maximum arity r, in which the length of an edge from a parent to its ith child is c_i where $c_1 \le c_2 \le \cdots \le c_r$ are r fixed positive reals. Figure 2 illustrates two finite lopsided trees, Figure 3 illustrates an infinite one. The name *lopsided trees* was only coined in 1989 by Kapoor and Reingold [19] but the trees themselves have been implicitly present in the literature at least since 1961 when Karp [20] used them to model minimum-cost prefix-free (Huffman) codes in which the length of the edge of the letters in the encoding alphabet were unequal; c_i represented the length of the ith letter in the encoding alphabet (the *idea* of such codes was already present in Shannon's seminal paper on communication theory [30]). Such trees were later used in [16] and [4] to design more efficient algorithms for the same problem.

For fixed $c_1 \le c_2 \le \cdots \le c_r$ we study three problems on these trees:

- Given n, characterize the combinatorial structure of a lopsided tree with n leaves that has minimal cost, where the cost of a tree is its external path length, i.e., the sum of the lengths of the paths from the root to all of the leaves.

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- Calculate exactly how many nodes of depth $\leq x$ exist in the infinite tree.
- Express the cost of the minimal cost tree as a function of n and the c_i.

To motivate these problems we first introduce the concept of a Varn code [32], [28]. Suppose that we wish to construct a prefix-free encoding of n symbols using an encoding alphabet of r letters, $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$ in which the length of character α_i is c_i, where the c_i's may all be different. As an example consider the Morse code alphabet $\Sigma = \{., -\}$ in which the length of a "dash" may be longer than that of a "dot." By a prefix-free encoding we mean a set of n strings $\{\omega_1, \ldots, \omega_n\} \subseteq \Sigma^*$ in which no ω_i is a prefix of any ω_j.

If a symbol is encoded using string $\omega = \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_l}$, then $cost(\omega) = \sum_{j \leq l} c_{i_j}$ is the length of the string. For example, if $r = 2$, $\Sigma = \{0, 1\}$, and $c_1 = c_2 = 1$, then the cost of the string is just the number of bits it contains. This last case is the basic one encountered in regular Huffman encoding [29], [12].

Now suppose that the n symbols to be encoded are known to occur with equal frequency. The *cost* of the code is then defined to be $\sum_{i \leq n} cost(\omega_i)$ (which divided by n is the average cost of transmitting or *length* of a symbol). Given $c_1 \leq c_2 \leq \cdots \leq c_r$ a *Varn code* for n symbols is a minimum-cost code. Varn codes have been extensively studied in the compression and coding literature; [28] contains a large bibliography and up-to-date description of what is currently known about them.

Such codes can be naturally modeled by lopsided trees in which the length of the edge from a node to its ith child is c_i; we call such an edge an ith *edge*. Suppose that v is a leaf in a lopsided tree and the unique path from the tree's root to v first traverses an i_1^{st} edge then an i_2^{nd} edge and so on up to an i_l^{th} edge. We can then associate with this leaf the codeword $\omega = \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_l}$. The cost of this codeword is exactly the same as the depth of v in the tree, i.e., $\sum_{j \leq l} c_{i_j}$. Using this correspondence, every tree with n leaves corresponds to a prefix-free set of n codewords and vice versa; the cost of the code is exactly equal to the external path length of the tree which we henceforth call the *cost* of the tree. This correspondence is extensively used, for example, in the analysis of Huffman codes. See Figure 1. A lopsided tree with minimal cost for n leaves is called an *optimal tree*. See Figure 2.

With this correspondence and notation we see that the problems of constructing a Varn code and calculating its cost are equivalent to those of constructing an optimal tree and calculating its cost. Under these two different guises these problems have been extensively studied in both the coding/compression and computer science communities. Algorithms for finding such trees appear in [19], [7], [32], [9], [26], and [18]. The first two citations are special cases; in [19] Kapoor and Reingold discuss the binary case ($r = 2$) and in [7] Choy and Wong address what they call α, β trees, trees in which r is fixed, and $\forall i$, $c_i = \alpha + (i-1)\beta$. Both special case algorithms run in $O(n)$ time. The remaining citations provide algorithms for the general case; the fastest one is [18] which runs in $O(n \log^2 r)$ time. The bottleneck in these general algorithms is that there are many possible trees with n leaves and all of the algorithms work by constructing restricted (but large) sets of such trees and somehow finding the minimal-cost one in the set; this tree will be the optimal one. (When $r = 2$ this restricted set will collapse to only one candidate tree; it is when $r > 2$ that problems arise.) Analyses of the costs of Varn codes or, equivalently, of the costs of optimal lopsided trees appear in [30], [23],

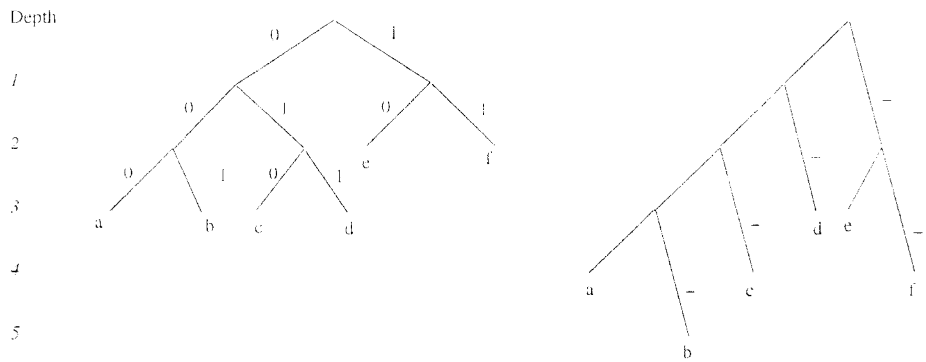


Fig. 1. Two trees with six leaves (labeled a, b, c, d, e, f). The tree on the left has $(c_1, c_2) = (1, 1)$. It corresponds to the prefix-free code

$$a = 000, \quad b = 001, \quad c = 011, \quad d = 011, \quad e = 10, \quad f = 11$$

for alphabet $\Sigma = \{0, 1\}$ when 0 and 1 have the same length. The cost of the code and corresponding tree is $3 + 3 + 3 + 3 + 2 + 2 = 13$. The tree on the right has $(c_1, c_2) = (1, 2)$. It corresponds to the prefix-free code

$$a = ..., \quad b = ...\_, \quad c = ..\_, \quad d = .\_, \quad e = \_., \quad f = \_\,\_$$

when $\Sigma = \{., \_\}$ and *length*$(.) = 1$, *length*$(\_) = 2$. The cost of the code and corresponding tree is $4 + 5 + 4 + 3 + 3 + 4 = 23$.

[10], [11], [2], [19], [28], and [1]. As the authors of these papers mention, their various analyses are only tight for some special cases but in most cases provide only loose upper and lower bounds. Many of the analyses use information theoretic techniques and therefore cannot be tight (since they do not fully model the tree). The most complete analysis is in [19] which derives a closed asymptotic expression but restricts itself to the binary-tree case for rational c_1/c_2, leaving both the binary irrational and the general r-ary cases open problems. Lopsided trees have also been used to model searching procedures

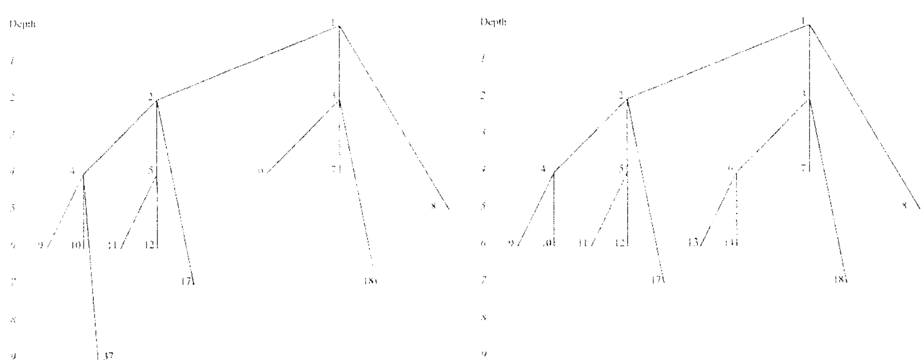


Fig. 2. The two trees pictured both have $r = 3$, $(c_1, c_2, c_3) = (2, 2, 5)$, and ten leaves each. The leftmost one has five internal nodes and cost 60; the rightmost one has six internal nodes and cost 59. We will see later that the rightmost one is optimal.

with r-ary branching in which the cost of discovering that the ith branch must be taken is c_i [31].

This paper has two main results, one combinatorial, the other analytic. Combining them yields a full analysis of the cost of Varn codes.

The first main result of this paper is a new combinatorial way of looking at optimal, i.e., mimimum-cost, lopsided trees. We assume that $c_1 \leq c_2 \leq \cdots \leq c_r$ are fixed and examine how the structure of optimal lopsided trees evolve as n increases. We prove that the trees evolve in a very regular and understandable fashion. This permits us to know what an optimal tree for n nodes looks like without having to search through a large collection of lopsided trees.

The second main result of this paper is an analysis of exactly how many nodes in the infinite tree have depth at most x. This problem reduces to analyzing Fibonacci-type recurrences of the form

$$L(x) = L(x - c_1) + L(x - c_2) + \cdots + L(x - c_r). \tag{1}$$

This problem is easy if the c_i are integers or rational multiples of each other, see, e.g., [33]. It gets complicated when the c_i are irrational. While the solution to the case $r = 2$ is implicit in the work of Fredman and Knuth [15] and, later, Pippenger [27], the general r-ary case does not seem to have been previously addressed. We show how to use Mellin transform and singularity analysis techniques to solve these equations. The solution to (1) will have a qualitatively different behavior depending upon whether the c_i are rational multiples of each other or not. This difference in behavior will be reflected in the ways in which the trees evolve as the number of leaves increase.

The analysis of $L(x)$ is of independent interest. We illustrate this by describing an application in the derivation of exact bounds on the time needed for broadcasting in the postal model of message passing in distributed computation, improving the bounds given in [3].

A more important application arises when we combine the analysis of $L(x)$ with the first result describing the structure of trees. This will yield the major result of this paper, an exact analysis of the cost of optimal trees or Varn codes. Because we know exactly how the structure of the optimal tree evolves as n grows we are able to calculate how the cost of the optimal tree increases with n. This provides, once and for all, a unified analysis that gives asymptotically exact bounds for Varn code costs in all cases.

The remainder of this paper is structured as follows. Section 2 motivates the problem and introduces the definitions that are used. Section 3 presents (mostly without proof) the three major results of this paper: (i) a description of the combinatorial structure of optimal trees; (ii) an asymptotic analysis of the solution to generalized Fibonacci recurrences and the subsequent analysis of the distribution of nodes in the infinite tree that it implies as well as the analysis of broadcasting protocols in the postal model that follows; (iii) an asymptotic analysis of optimal Varn lopsided trees or, equivalently, Varn codes.

Sections 4–6 prove the respective results of parts (i)–(iii).

We point out that our first result, the derivation of how the combinatorial structure of the trees evolves, can be used to design a new algorithm for constructing optimal lopsided trees in $O(n \log r)$ time, beating the old $O(n \log^2 r)$ bound. This algorithm and its analysis are presented in a companion paper [6].

2. Definitions. In this section we motivate and introduce the definitions used in the rest of the paper. In what follows $\{x\} = x - \lfloor x \rfloor$ is the *fractional part* of x, e.g., $\{7.32\} = 0.32$.

Now let $0 < c_1 \leq c_2 \leq \cdots \leq c_r$ be r fixed reals.

DEFINITION 1. The **infinite *r*-ary tree** is the infinite, rooted, r-ary tree such that the length of the edge connecting a node to its ith child is c_i. See Figure 3.

A **lopsided tree** is a subtree T of the infinite r-ary tree containing the root. See Figure 4.

If u is a node in the infinite tree, then $\mathbf{child_i(u)}$ is the ith child of u, e.g., in Figure 3 $child_1(4) = 8$, $child_2(4) = 12$, and $child_3(4) = 18$.

In standard trees the *depth* of a node v is defined to be the number of edges on the unique path from the root to v. The *external path length* of the tree is the sum of the depth of all external nodes. This definition can easily be extended to lopsided trees if we redefine the depth of v to be the sum of the lengths of the edges on the path connecting the root to v.

DEFINITION 2. Let u be a node and let T be a lopsided tree.

depth(*u*) is the sum of the lengths of the edges on the path connecting the root to u.

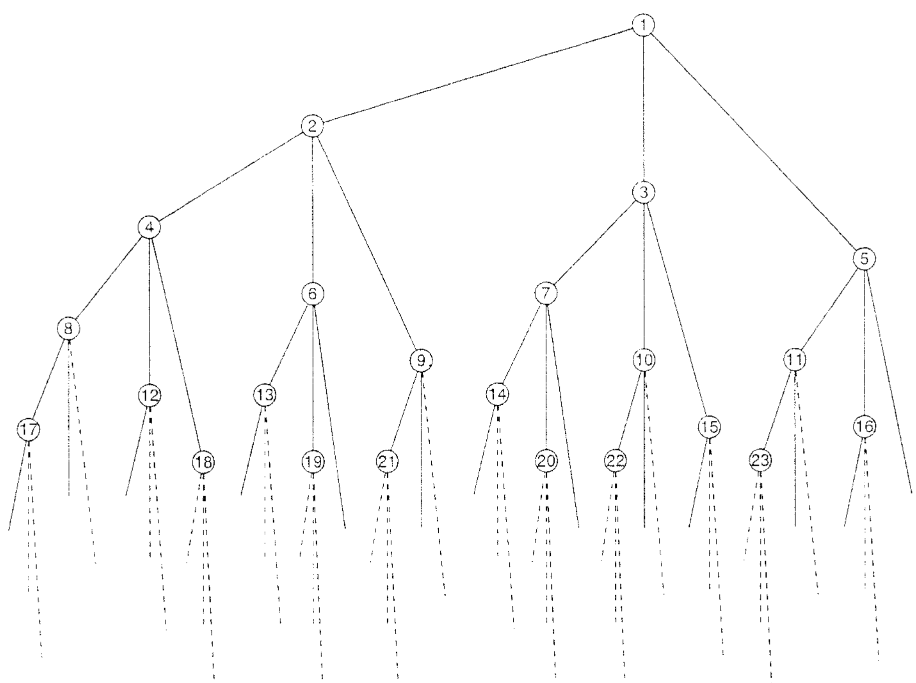


Fig. 3. The labeled infinite 3-ary tree with $(c_1, c_2, c_3) = (3, 5, 7)$. Nodes are drawn so that the depth of nodes on the page corresponds to their depths in the tree. This convention is followed throughout this paper.

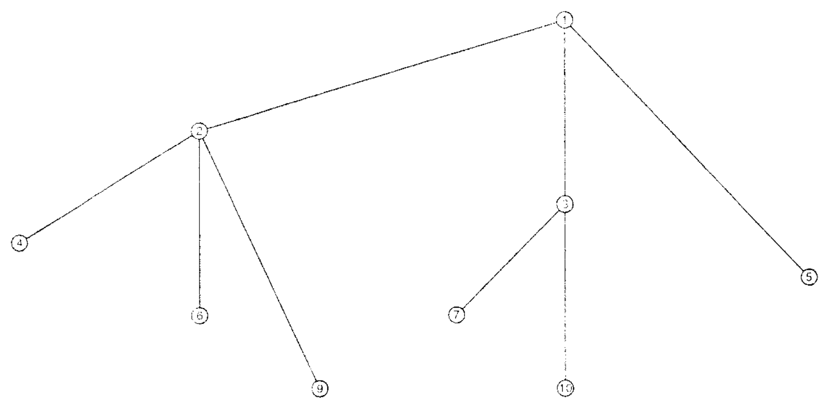


Fig. 4. A lopsided tree with $r = 3$ and $(c_1, c_2, c_3) = (3, 5, 7)$.

The **external path length** or **cost** of T is $C(T) = \sum_{v \text{ a leaf of } T} depth(v)$, the sum of the depths of all external nodes.

The **height** of T is $H(T) = \max_{u \in T} depth(u)$.

For example, in the tree T in Figure 4, $depth(4) = c_1 + c_1 = 3 + 3 = 6$, $depth(6) = c_1 + c_2 = 3 + 5 = 8$, and $C(T) = 6 + 8 + 10 + 8 + 10 + 7 = 49$. $H(T) = 10 = depth(9) = depth(10)$.

DEFINITION 3. A tree T with n leaves is **optimal** if it has the minimal cost among all lopsided trees with n leaves. We denote such an optimal tree by T_n (note that it might not be unique). See Figure 5.

In our analysis of the structure of the trees we are interested in knowing at what depths nodes can appear in the infinite tree. It is obvious that nodes will appear at exactly the set of depths $\{\sum_{i=1}^{r} a_i c_i\}$ where the a_i range over all nonnegative integers. For example, if $(c_1, c_2) = (15, 25)$, then nodes can only have depths that are multiples of $5 = \gcd(15, 25)$ and, deep enough in the tree, nodes will appear on *every* level with depth a multiple of 5. If, though, $(c_1, c_2) = (3, \pi)$, then nodes appear on all depths that can be written in the form $a_1c_1 + a_2c_2$, $a_1, a_2 \geq 0$ integers, and general theorems about irrational numbers [25] imply that the depth difference between (neighboring) successive levels upon which nodes appear tends to zero. This is discussed in more detail in Section 3.2. To formalize this distinction we introduce the following definitions:

DEFINITION 4. Let $(c_1, \ldots, c_r)$ be a tuple of r positive reals:

1. The tuple is *rationally related* if there exists $d > 0$ and positive integers $(c_1', \ldots, c_r')$ such that

$$(c_1, \ldots, c_r) = d \cdot (c_1', \ldots, c_r') \qquad \text{and} \qquad \gcd(c_1', \ldots, c_r') = 1.$$

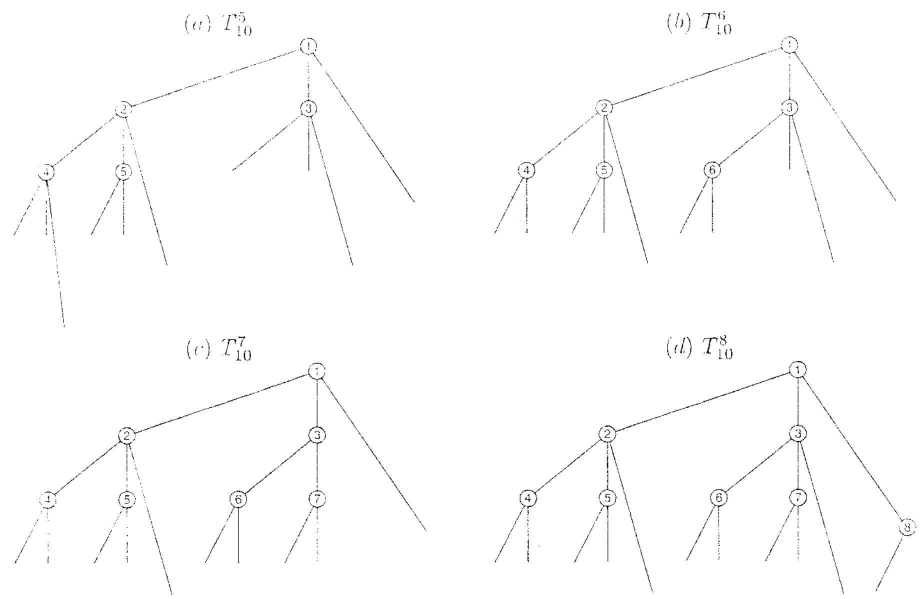


Fig. 5. For $r = 3$, $(c_1, c_2, c_3) = (2, 2, 5)$. The cost of the trees are, respectively, 60, 59, 60, 62. It can be shown that (b) is an optimal tree for $n = 10$ leaves.

2. If $(c_1, \ldots, c_r)$ is rationally related we define the gcd of the tuple by $\gcd(c_1, \ldots, c_r) = d$.
3. If $(c_1, \ldots, c_r)$ is not rationally related it is said to be *irrationally related*.

For example, triple $(2\pi, 4\pi, 6\pi)$ is rationally related because $(2\pi, 4\pi, 6\pi) = 2\pi(1, 2, 3)$ but the triple $(1, 4, \pi)$ is not rationally related. Note that if the $(c_1, \ldots, c_r)$ are all integers, then the gcd defined above is the standard *greatest common divisor*, e.g., $(10, 20, 35) = 5(2, 4, 7)$ so $\gcd(10, 20, 35) = 5$.

There is one more definition that we need. It turns out that optimal trees of a certain size have a bottom "fringe" of size h where h is defined in the following lemma (which is proved in Section 4):

LEMMA 1. *Let $x_m = (\sum_{i=1}^{m} c_i)/(m-1)$ for $m = 2, \ldots, r$. There exists $k \geq 2$ such that*

$$x_2 \geq x_3 \geq \cdots \geq x_{k-1} \geq x_k < x_{k+1} < \cdots < x_r. \tag{2}$$

(If $x_2 \leq x_3$, set $k = 2$. If $x_2 \geq x_3 \geq \cdots \geq x_{r-1} \geq x_r$, set $k = r$.) Letting k be this value and setting $h \stackrel{\text{def}}{=} x_k$ we have, further, that, if $k < r$, then $c_k \leq h < c_{k+1}$.

3. The Results. In this section we present the major results of this paper. Proofs of most of the results are deferred until later.

Before starting the discussion of *lopsided* trees we try to give some intuition by quickly reviewing what is known about the standard, *nonlopsided*, tree, i.e., $r = 2$ and

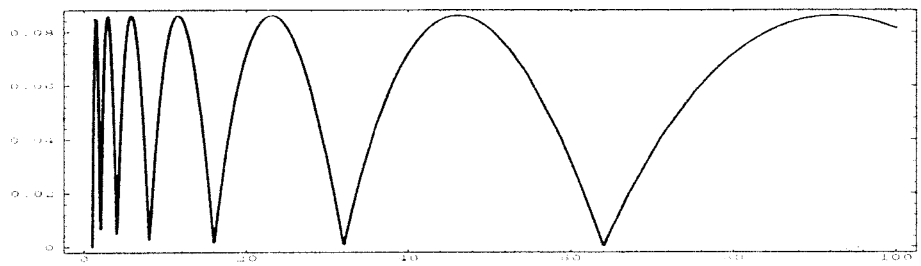


Fig. 6. The function $B(\log_2 n) = 2 - \{\log_2 n\} - 2^{1-\{\log_2 n\}}$.

$c_1 = c_2 = 1$. It is well known, e.g., [22, 5.3.1], that such a tree with minimal external path length for n leaves must have the following property: all leaves must appear on level $l = \lfloor \log_2 n \rfloor$ or level $l + 1$. This means that there will be $2^{l+1} - n$ leaves on level l and $2n - 2^{l+1}$ leaves on level $l + 1$. The external path length is then

$$l(2^{l+1} - n) + (l + 1)(2n - 2^{l+1}) = ln + 2n - 2^{l+1}.$$

The question now is how to rewrite this as a function of n. The standard way of doing so is to note that

$$l = \log_2 n - \{\log_2 n\}, \qquad 2^{l+1} = n2^{1-\{\log_2 n\}}.$$

The external path length can therefore be rewritten as

$$\begin{aligned} ln + 2n - 2^{l+1} &= n \log_2 n - n\left[2 - \{\log_2 n\} - 2^{1-\{\log_2 n\}}\right] \\ &= n \log_2 n + nB(\log_2 n), \end{aligned}$$

where $B(\theta) = 2 - \{x\} - 2^{1-\{x\}}$ is periodic with period 1, i.e., $B(1 + x) = B(x)$. See Figure 6.

Two important observations to keep in mind about this example are: (i) the optimal tree was one that tried to keep the leaves as "balanced" as possible; (ii) the fact that the leaves can occur on two distinct levels a unit distance apart introduced the periodic $B(\theta)$ term into the expression for path length. This periodic term "corrects" for the fact that there is a discrete jump between successive levels. Both of these observations prove helpful in understanding the results in the remainder of this section.

3.1. *The Structure of Optimal Trees*. In this subsection we describe trees T_n that have minimal external path length among all trees with n leaves. More particularly we examine how the structure of the optimal trees changes as n grows.

We start by labeling the nodes of the infinite tree as $1, 2, 3, \ldots,$ in order of increasing depth, breaking ties arbitrarily. That is, if u and v are two nodes with $depth(u) < depth(v)$, then $u < v$; if $depth(u) = depth(v)$, break ties arbitrarily. Figures 3 and 7 provide examples of such labelings.

DEFINITION 5. Let V be any set of nodes. Set

$$LEAF(V) = \{u\colon\ parent(u) \in V, u \notin V\}.$$

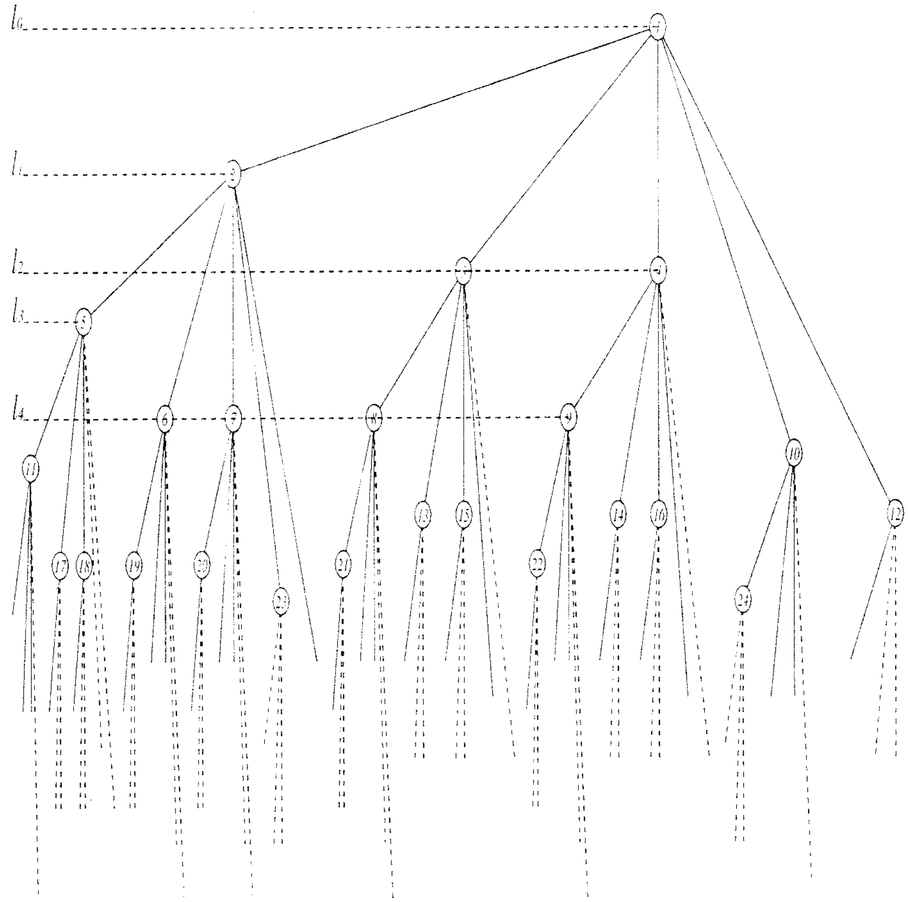


Fig. 7. The top of the infinite tree with $(c_1, c_2, c_3, c_4, c_5) = (3, 5, 5, 8.75, 10)$. $(r = 5)$. Note that $l_0, l_1, l_2, l_3, l_4, \ldots = 0, 3, 5, 6, 8 \ldots$ and $m_0, m_1, m_2, m_3, m_4 = 1, 2, 4, 5, 9$.

For $n \le |LEAF(V)|$, let

$$LEAF_n(V) = \text{the } n \text{ smallest labeled nodes in } LEAF(V).$$

$LEAF_n(V)$ is the set of n smallest labeled nodes that are children of nodes in V but are not in V themselves. $LEAF_n(V)$ contains n highest (smallest depth) in $LEAF(V)$.

For example, in Figure 7, $LEAF_5(\{1, 2, 3\}) = \{4, 5, 6, 7, 8\}$.

It is obvious that an optimal tree must be *proper* (each of its internal nodes must have at least two children) otherwise some internal node can be replaced by its child, decreasing the cost. This property of being *proper* will be useful later.

For any given n, let m be the number of internal nodes of some proper tree having n leaves. The total number of edges in the tree is $n + m - 1$. Since every internal node has between two and r children, $2m \le n + m - 1 \le rm$, or $\lceil (n-1)/(r-1) \rceil \le m \le n - 1$.

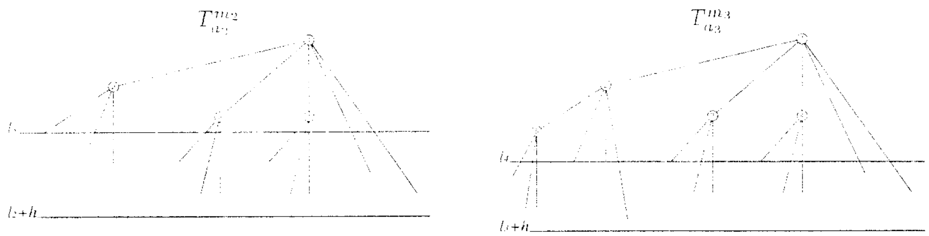


Fig. 8. In the first case of Theorem 1 when $n = a_j$, the optimal tree $T_n = T_{a_j}^{m_j}$. In this example, $a_2 = 11$, $a_3 = 14$. Therefore $T_{11} = T_{a_2}^{m_2}$ and $T_{14} = T_{a_3}^{m_3}$.

DEFINITION 6. Let $\lceil (n-1)/(r-1) \rceil \leq m \leq n-1$, and $V_m = \{1, 2, \ldots, m\}$. Set $T_n^m = V_m \cup LEAF_n(V_m)$.

By definition, if $u \in T_n^m$ is not a root, then $parent(u) \in T_n^m$ so T_n^m is a tree. These trees are called *shallow trees* in [18] which uses the simple observation that there must be some shallow tree that is also optimal as the basis of an algorithm for constructing optimal trees. Figure 5 illustrates the shallow trees T_{10}^5, T_{10}^6, T_{10}^7, and T_{10}^8 when $(c_1, c_2, c_3) = (2, 2, 5)$.

Return now to the infinite tree and let $l_0, l_1, l_2, \ldots,$ be the consecutive levels upon which nodes appear, i.e., $l_0 = 0$ and

$$\forall i > 0, \qquad l_i = \min\{depth(v)\colon\ v \text{ a node with } depth(v) > l_{i-1}\}.$$

Thus, for example, $l_1 = c_1$, $l_2 = \min(2c_1, c_2)$, etc.

Also let m_j be the number of nodes higher than or on depth l_j (Figure 7):

$$m_j = |\{v \text{ a node: } depth(v) \leq l_j\}|.$$

With these definitions, we can now state our main combinatorial result. Figures 8–11 illustrate the various cases of the theorem for $r = 5$ and $(c_1, c_2, c_3, c_4, c_5) = (3, 5, 5, 8.75, 10)$. The proof of the theorem is given in Section 4.

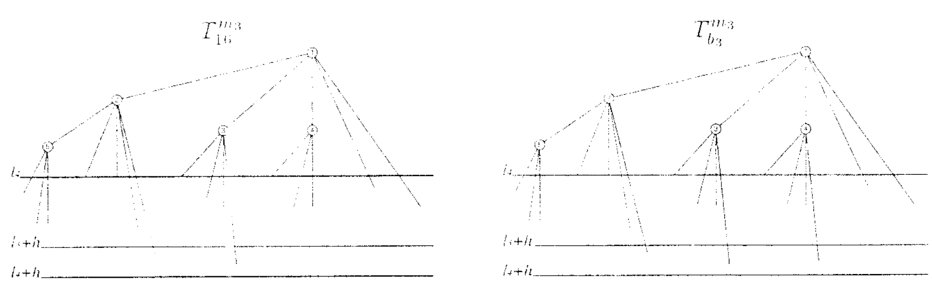


Fig. 9. In the second case of Theorem 1 when $a_j < n \leq b_j$, the optimal tree $T_n = T_n^{m_j}$. In this example, $a_3 = 14$, $b_3 = 17$. So $T_{15} = T_{15}^{m_3}$ (not pictured), $T_{16} = T_{16}^{m_3}$, and $T_{17} = T_{17}^{m_3}$.

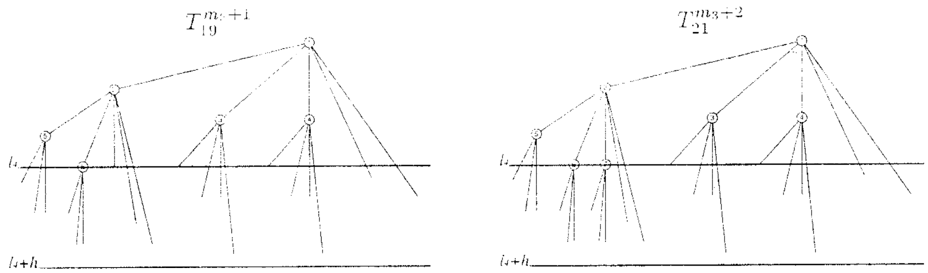


Fig. 10. In the third case of Theorem 1 $b_j < n < a_{j+1}$. If $n = b_j + p(k-1)$, the optimal tree for n leaves is $T_n = T_n^{m_j+p}$. In this example, $b_3 = 17$, $19 = 17 + 1 \cdot (3-1)$, and $21 = 17 + 2 \cdot (3-1)$. So $T_{19} = T_{19}^{m_3+1}$ and $T_{21} = T_{21}^{m_3+2}$.

THEOREM 1. *Given $c_1 \le c_2 \cdots \le c_r$ let k and h be as defined in Lemma* 1 *and let l_j and m_j be as defined above. Set*

$$A_j = \{v \in LEAF(V_{m_j}) \colon depth(v) \le l_j + h\}$$

and $a_j = |A_j|$, the number of nodes in A_j, set

$$B_j = A_j \cup \{v \in LEAF(V_{m_j}) \colon l_j + h < depth(v) \le l_{j+1} + h\}$$

and $b_j = |B_j|$, the number of nodes in B_j.

1. *If $n = a_j$, then the tree $T_{a_j}^{m_j} = V_{m_j} \cup A_j$ is optimal.*
2. *If $a_j < n \le b_j$, then the tree $T_n^{m_j}$ is optimal; furthermore, $T_{b_j}^{m_j} = V_{m_j} \cup B_j$.*
3. *If $b_j < n \le a_{j+1}$, then*
   (a) *$n = b_j + p(k-1)$, then $T_n^{m_j+p}$ is optimal;*
   (b) *$n = b_j + p(k-1) + q$ with $q < k-1$, then one of $T_n^{m_j+p}$ and $T_n^{m_j+p+1}$ is optimal.*

Intuitively this theorem reverses the problem. Instead of asking "how many internal

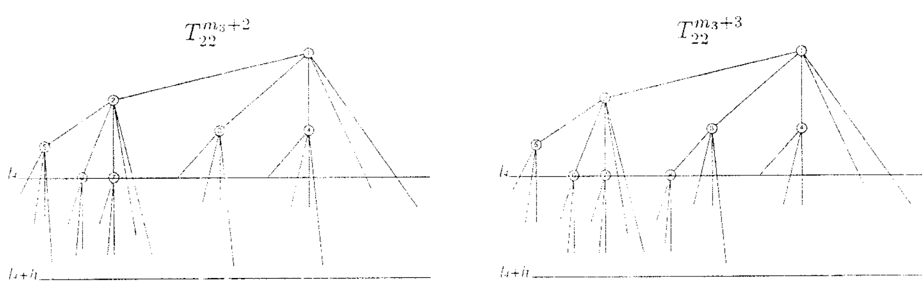


Fig. 11. In the second part of the third case of Theorem 1 $b_j < n < a_{j+1}$ and $n = b_j + p(k-1) + q$ with $0 < q < k-1$, the optimal tree for n leaves is either $T_n^{m_j+p}$ or $T_n^{m_j+p+1}$. In this example, $b_3 = 17$, $22 = 17 + 2 \cdot (3-1) + 1$. The cost of $T_{22}^{m_3+2}$ is 246.75 while the cost of $T_n^{m_3+3}$ is 247.25. So $T_{22} = T_{22}^{m_3+2}$.

nodes are in an optimal tree with n leaves?" it instead asks, and answers, the question "if T is an optimal tree with m internal nodes, how many leaves n_m can T have?"

We now briefly sketch the theorem's implications. Figures 8–11 illustrate the discussion. To build tree $T_{a_j}^{m_j}$ draw a horizontal line across the infinite tree at depth l_j; the nodes on or above this line are the m_j nodes in V_{m_j}, the internal nodes in $T_{a_j}^{m_j}$. Next draw a second horizontal line at depth $l_j + h$. The nodes in V_{m_j} have exactly a_j children on or above this line. These nodes, the A_j, will be the leaves of $T_{a_j}^{m_j}$. The tree $T_{a_j}^{m_j}$ will be optimal.

Now draw a third horizontal line at depth $l_{j+1} + h$. There are $b_j - a_j$ children of the V_{m_j} between the second and third lines (actually, below the second and on or above the third). If n satisfies $a_j < n \le b_j$, then construct $T_n^{m_j}$ by taking the $n - a_j$ highest nodes between the second and third lines and adding them to $T_{a_j}^{m_j}$ as leaves. The tree $T_n^{m_j}$ will be optimal. The largest such tree constructed this way is $T_{b_j}^{m_j}$.

If $b_j < n \le a_{j+1}$ the theorem says that optimal T_n^m can be constructed as follows. If n is of the form $n = b_j + p(k-1)$ take the p highest leaves in $T_{b_j}^{m_j}$, make them internal, and add their k highest children to the tree.

If $b_j < n < a_{j+1}$ but $n = b_j + p(k-1) + q$ with $0 < q < k-1$, then we do not know a priori what the optimal tree must be. By looking at the definitions we do know, though, that it must be either the tree that results from starting with $T_{b_j+p(k-1)}^{m_j+p}$ and adding the q smallest unused leaves in V_{m_j+p} to the tree or starting with $T_{b_j+(p+1)(k-1)}^{m_j+p+1}$ and erasing the $k - 1 - q$ deepest leaves in that tree.

3.2. *Growth of the Infinite Tree.* We need to understand how the infinite tree evolves as its depth grows. Set $A_x = \{v \text{ a node} : depth(v) \le x\}$ to be the tree containing all nodes of depth at most x in the infinite tree. Then set

$$F(x) = \text{ number of nodes in } A_x, \qquad L(x) = \text{number of leaves in } A_x.$$

Tree A_x has a root and subtrees falling off of each of its r children. The subtree falling off of the ith child has the same structure as A_{x-c_i} so (Figure 12) the equations satisfy

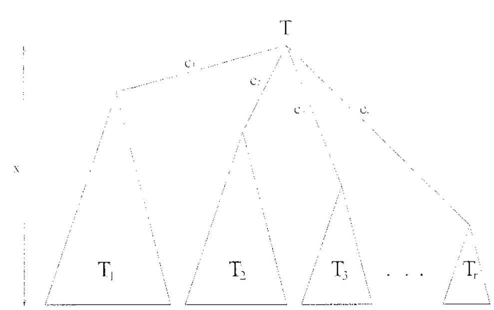


Fig. 12. Tree T_i is the subtree rooted at the ith child of the root of T. If T has height x, then T_i has height $x - c_i$ and contains $F(x - c_i)$ nodes.

the Fibonacci-type recurrences

(3)
$$F(x) = \begin{cases} 1 + F(x - c_1) + F(x - c_2) \\ \qquad + \cdots + F(x - c_r), & \text{if} \quad x \geq c_1; \\ 1, & \text{if} \quad 0 \leq x < c_1; \\ 0, & \text{if} \quad x < 0; \end{cases}$$

and

(4)
$$L(x) = \begin{cases} L(x - c_1) + L(x - c_2) + \cdots + L(x - c_r), & \text{if} \quad x \geq c_1; \\ 1, & \text{if} \quad 0 \leq x < c_1; \\ 0, & \text{if} \quad x < 0. \end{cases}$$

Both of these equations are defined for all real x.

Now notice that if a node $u \in A_x$ has *depth*$(u) > x - c_1$, then any child of u will have depth greater than x; u therefore can have no children and is a leaf. If *depth*$(u) \geq x - c_1$, then at least the first child of u is in A_x and u is not a leaf. We have just shown that u is a leaf if and only if *depth*$(u) > x - c_1$ and thus

(5)
$$L(x) = F(x) - F(x - c_1).$$

We will now see that the behavior of the solution of $F(x)$, and thus $L(x)$ as well, will depend very strongly upon whether the c_i are rationally related or not. This qualitative difference will strongly influence the analysis of optimum lopsided trees.

Suppose that the $(c_1, \ldots, c_r)$ are given. A node can exist on level x of the infinite tree if and only if there is a path of length x in the tree; this in turn only happens if there exist nonnegative integers $a_1, \ldots, a_r \geq 0$ such that $\sum_i a_i c_i = x$. The existence of such integers corresponds to the existence of a path containing exactly a_1 first edges from a nodes, a_2 second edges, etc. The number of nodes on level x will be the number of paths that can be built using a_1 first edges, a_2 second edges, etc., which is

$$\sum_{\substack{a_1 c_1 + \cdots + a_r c_r = x \\ a_1, \ldots, a_r \geq 0}} \binom{a_1 + a_2 + \cdots + a_r}{a_1,\ a_2,\ \ldots,\ a_r},$$

where the summation is over all tuples $(a_1, a_2, \ldots, a_r)$ satisfying the conditions. Thus

(6)
$$F(x) = \sum_{\substack{a_1 c_1 + \cdots + a_r c_r \leq x \\ a_1, \ldots, a_r \geq 0}} \binom{a_1 + a_2 + \cdots + a_r}{a_1,\ a_2,\ \ldots,\ a_r}.$$

Suppose now that $(c_1, c_2) = (3, 5)$. Then $F(x)$ can only change at integer values of x that can be written in the form $3a_1 + 5a_2$, i.e., $x = 1, 3, 5, 6, 8, 9, 10 \ldots$ and every integer $x \geq 10$. The fact that for $x \geq 8$ the difference between successive depths upon which nodes appear is a constant $(= 1)$ is not unique to this pair of c_i; for all c_1, c_2 with $\gcd(c_1, c_2) = 1$ there exists some integer N such that for every integer $n \geq N$, there exist $a_1, a_2 \geq 0$ with $a_1 c_1 + a_2 c_2 = n$. In this case $F(x)$ changes exactly at integer values and remains unchanged between them.

On the other hand if $(c_1, c_2) = (3, \pi)$, then successive levels of the form $a_1c_1 + a_2c_2$ tend to be closer and closer together with the difference between successive levels tending to zero.

More generally if $(c_1, \ldots, c_r)$ are rationally related, then the infinite tree can only have nodes on levels that are integer multiples of $d = \gcd(c_1, \ldots, c_r)$; furthermore, there is some X such that if $x \geq X$ is an integral multiple of d, then the tree contains nodes on level x. By contrast, if $(c_1, \ldots, c_r)$ are irrationally related we will now see that the distance between successive levels upon which nodes appear will decrease to zero. This is a consequence of the Kronecker–Weyl theorem on the uniform distribution of the multiples of irrational numbers [25]. (Pippenger [27] gives a more sophisticated application of the Kronecker–Weyl theorem to Diophantine combinations of irrationally related pairs.) We actually do not need the full power of the Kronecker–Weyl theorem; we only use the fact that if θ is an irrational number, then the sequence $\{\theta\}, \{2\theta\}, \{3\theta\}, \{4\theta\}, \ldots$ is *dense* in [0, 1].

Suppose that $(c_1, \ldots, c_r)$ is irrationally related. This implies that there are some c_i, c_j such that c_i/c_j is irrational. We will now show that even the levels in $\{a_ic_i + a_jc_j, : a_i, a_j \geq 0\}$, the set that can be reached using only c_i and c_j edges, have the property that the distance between successive levels upon which nodes appear will decrease to zero. Without loss of generality we may scale by dividing by c_1 and assume that $c_i = 1$ and $c_j = \theta$ is irrational (scaling maintains the property that the differences go to zero). Let $U = \{a + b\theta \colon a, b \geq 0\}$ be the set of level depths. Fix m and, $\forall n \geq 0$, $\forall 0 \leq t < m$, define

$$U_{n,t} = \left[n + \frac{t}{m},\, n + \frac{t+1}{m} \right] \cap U.$$

We will show that for all n large enough and all t, $U_{n,t} \neq \emptyset$. Since this will be true for every m it will prove that the distance between successive levels upon which nodes appear will decrease to zero.

Note that if $x = a + b\theta \in U$, then $x + 1 = (a + 1) + b\theta \in U$. Thus, if for some n' and specific t, $U_{n',t} \neq \emptyset$, then, $\forall n > n'$, $U_{n,t} \neq \emptyset$. From the fact that $\{\theta\}, \{2\theta\}, \{3\theta\}, \{4\theta\}, \ldots$ is dense in [0, 1] we know that there is some N such that, $\forall 0 \leq t < m$, there is some element of $\{\theta\}, \{2\theta\}, \{3\theta\}, \{4\theta\}, \ldots, \{N\theta\}$ in each of the subintervals $[t/m, (t + 1)/m]$. This, then, immediately implies that, $\forall n > \lceil N\theta \rceil$ and $\forall 0 \leq t < m$, $U_{n,t} \neq \emptyset$ and we are done.

We encapsulate the above comments in a lemma for later use:

LEMMA 2. *Let $(c_1, \ldots, c_r)$ be an r-tuple of nonnegative reals.*

- *If $(c_1, \ldots, c_r)$ is rationally related with $d = \gcd(c_1, \ldots, c_r)$, then $\exists J$ such that, $\forall j > J$, $l_{j+1} - l_j = d$.*
- *If $(c_1, \ldots, c_r)$ is irrationally related, then* $\lim_{j\to\infty}(l_{j+1} - l_j) = 0$.

We now state the exact asymptotics of $F(x)$.[4] Note that they reflect the qualitative difference described above. The proof of this theorem is given in Section 5.

[4] The asymptotics of $F(x)$ for $r = 2$ can be derived from [15] and [27] which, in different ways, both study the function $h(x) = F(\ln x)$.

THEOREM 2. *Let $(c_1, \ldots, c_r)$ be an r-tuple of nonnegative reals and define*

$$F(x) = \begin{cases} 1 + F(x - c_1) + F(x - c_2) & \\ \qquad + \cdots + F(x - c_r), & \text{if} \quad x \geq c_1; \\ 1, & \text{if} \quad 0 \leq x < c_1; \\ 0, & \text{if} \quad x < 0. \end{cases} \tag{7}$$

Let α be the smallest real positive root of the equation $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$ and $\varphi = 1/\alpha$. Let $c = \sum_{i=1}^{r} c_i \varphi^{-c_i})$. Then:

1. *If $(c_1, \ldots, c_r)$ is rationally related,*

$$F(x) = D(x)\varphi^x + O(\rho^x),$$

where $D(x) = (d/c(1 - \varphi^{-d}))\varphi^{-d\{x/d\}}$ is a periodic function with period d and $0 \leq \rho < \varphi$.

2. *If $(c_1, \ldots, c_r)$ is irrationally related,*

$$F(x) = \frac{1}{c \ln \varphi}\varphi^x + o(\varphi^x).$$

If the $(c_1, \ldots, c_r)$ are rationally related, then $F(x)$ only changes at integral multiples of d; if the $(c_1, \ldots, c_r)$ are irrationally related, then, as x increases, the difference between values at which $F(x)$ changes gets smaller and smaller so $F(x)$ behaves more and more as a continuous function.

We can plug the results of the theorem into (5) to find the behavior of $L(x)$. We summarize the results in the next theorem.

THEOREM 3. *Let $(c_1, \ldots, c_r)$ be an r-tuple of nonnegative reals and define*

$$L(x) = \begin{cases} L(x - c_1) + L(x - c_2) + \cdots + L(x - c_r), & \text{if} \quad x \geq c_1; \\ 1, & \text{if} \quad 0 \leq x < c_1; \\ 0, & \text{if} \quad x < 0. \end{cases} \tag{8}$$

Let α be the smallest real positive root of the equation $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$ and $\varphi = 1/\alpha$. Let $c = \sum_{i=1}^{r} c_i \varphi^{-c_i})$. Then:

1. *If $(c_1, \ldots, c_r)$ is rationally related,*

$$L(x) = E(x)\varphi^x + O(\rho^x),$$

where $E(x) = (d/c(1 - \varphi^{-d}))\varphi^{-d\{x/d\}}1 - \varphi^{-c_1})$ is a periodic function with period d and $\rho < \varphi$.

2. *If $(c_1, \ldots, c_r)$ is irrationally related,*

$$L(x) = \frac{1 - \varphi^{-c_1}}{c \ln \varphi}\varphi^x + o(\varphi^x).$$

Note: In the rational case we use the fact that c_1 is an integral multiple of d to find that $D(x - c_1) = D(x)$.

Theorems 2 and 3 look rather strange; they seem to imply that the rational and irrational cases have totally different behaviors. This is actually not so. The different behaviors of the rational and irrational case actually reflect the fact that they have the same instantaneous rate of growth. Since this is tangential to the focus of this paper we have removed this explanation to the Appendix.

3.2.1. *The Minimum Height of a Tree.* In this subsection we derive the the minimum height of a tree with n leaves and discuss basic applications in the theory of distributed broadcasting protocols. [5]

Denote the minimum height of a tree with n leaves as $H(n)$; it is given by the recurrence

$$H(n) = \min_{\substack{n_1+n_2+\cdots+n_r=n \\ n_1,n_2,\ldots,n_r\geq 0}} \left[\max_{n_i\neq 0}(H(n_i)+c_i)\right], \tag{9}$$

where $H(1) = H(0) = 0$. Instead of attempting to solve this equation directly we examine the structure of the minimum-height trees by using the results of the previous subsection.

First notice that $H(n)$ is certainly a nondecreasing function because if $H(n) = m$, then there is a tree T of height m containing n leaves. Peeling away leaves from the tree until there are $n-1$ leaves yields a tree with height at most m, so $H(n-1) \leq m$ (note that we might have to peel away many leaves because some might be the only child of their parent and discarding them does not decrease the number of leaves).

The tree containing all nodes of depth at most x has $L(x)$ leaves so $H(L(x)) \leq x$. Suppose x is a depth at which some nodes exist. It would be convenient if we could say that $H(L(x)) = x$. Unfortunately this is not always true. Consider, for example, the case in which $r = 2$, $c_1 = 1$, and $c_2 = 2$. Let $x = n$ for some large integer n and examine the tree A_n containing all nodes of depth at most n. There is only one node at depth n; the node reached by traversing n left (c_1) edges. The sibling of this node is at depth $x + \pi - 1 > \pi$ and so is not in the tree. Removing the node at depth x therefore leaves a tree with $L(x)$ leaves and depth slightly less than x so $H(L(x)) < x$.

We use the following observation instead. Returning to the recurrence relation (4) defining $L(x)$ we see that the same recurrence relation also describes the *maximum* number of leaves that a tree of height x can have. This implies that if $n > L(x)$, then $H(n) > x$. Using this fact we can derive the asymptotic behavior of $H(n)$:

THEOREM 4. *Let* $(c_1, \ldots, c_r)$ *be an R-tuple of nonnegative reals and let* $H(n)$ *be defined as above. Let* α *be the smallest real positive root of the equation* $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$, $\varphi = 1/\alpha$ *and* $c = \sum_{i=1}^{r} c_i\varphi^{-c_i}$).

1. *If* $(c_1, \ldots, c_r)$ *is rationally related with* $d = \gcd(c_1, \ldots, c_r)$, *then*

$$H(n) = d\left\lceil \frac{1}{d}\left(\log_\varphi n - \log_\varphi\left(\frac{d(1-\varphi^{-c_1})}{c(1-\varphi^{-d})}\right)\right)\right\rceil + O(1).$$

[5] These applications are gone into in more detail in a companion paper [17] which applies some of the techniques developed in parts of this article to analyze generalizations of lopsided trees that model various broadcasting protocols. This section is the jumping off point for [17].

Furthermore, there is some $N > 0$ *such that if* $n > N$, *then the* $O(1)$ *term is of the form* $+d$, 0, *or* $-d$.

2. *If* $(c_1, \ldots, c_r)$ *is irrationally related,*

$$H(n) = \log_\varphi n - \log_\varphi \frac{1 - \varphi^{-c_1}}{c \ln \varphi} + o(1).$$

PROOF. We first deal with the case that $(c_1, \ldots, c_r)$ are rationally related. Recall that nodes can only appear at levels of the tree that are integral multiples of d. Substituting $x = md$ into Theorem 3 yields

$$L(md) = \frac{d(1 - \varphi^{-c_1})}{c(1 - \varphi^{-d})} \varphi^{md} + o(\varphi^{md}). \tag{10}$$

This implies that there is some integer M such that, for all $m > M$, $L(md - d) < L(md)$. Therefore if n is large enough there must be an integer m such that $L(md - 1) < n \le L(md)$ implying $H(n) = md$. Inverting (10) completes the proof of part 1.

In the irrational case, for fixed $\varepsilon > 0$, Theorem 3 implies that there is some X such that $L(x - \varepsilon) < L(x)$ for all $x > X$. This in turn implies that if n is large enough we can find x such that $L(x - \varepsilon) < n \le L(x)$ and therefore that $x - \varepsilon < H(n) \le x$. Inverting $L(x) = ((1 - \varphi^{-c_1})/(c \ln \varphi))\varphi^x + o(\varphi^x)$ yields part 2 of the theorem. □

3.2.2. *Applications*. In [3] Bar-Noy and Kipnis introduce the *postal model* of message passing for distributed systems. In this model, counting time from when a sender first starts sending a message, the sender requires one unit of time before completing the work of sending and being able to do something else but the recipient requires λ units of time to receive and process it: λ is a parameter representing the *latency* of the system. Bar-Noy and Kipnis demonstrated that, in t time units, the maximum number of recipients that can receive a broadcast message in a one-to-many broadcast scheme in this model satisfies

$$F_\lambda(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le \lambda, \\ F_\lambda(t-1) + F_\lambda(t-\lambda) & \text{if} \quad t \ge \lambda. \end{cases} \tag{11}$$

The minimum time that it takes to broadcast to n recipients in the model therefore satisfies $f_\lambda(n) = \min\{t\colon F_\lambda(t) \ge n\}$.

Notice, though, that $F_\lambda(t)$ is exactly the function that we have labeled $L(t)$ with parameters $c_1 = 1$, $c_2 = \lambda$. The recurrence relations are exactly the same; to check that the initial conditions match it is enough to note that, for $1 \le t < \lambda$, $L(t) = L(t-1)$ so $L(t) = 1$ for $0 \le t < \lambda$. Therefore $f_\lambda(x) = H(x)$. Applying Theorem 4 yields

$$f_\lambda(n) = \log_\varphi n + O(1),$$

where $\varphi = 1/\alpha$, where α is the smallest positive root of $1 - z - z^\lambda$.

This improves the upper bound of

$$f_\lambda(n) \le 2\lambda + \frac{2\lambda \log n}{\log(\lceil\lambda\rceil + 1)}$$

given in [3]. The amount of improvement depends upon the particular value of λ but, as noted in [17], the ratio of the old bound over the new one goes to

$$\frac{2\lambda \log n}{\log(\lceil\lambda\rceil + 1)} \frac{1}{\log_\varphi n} = \frac{2\lambda}{\log_\varphi(\lceil\lambda\rceil + 1)} \to 2$$

as $\lambda \to \infty$.

3.3. *The Cost of Varn codes*. Combining Theorems 2 and 1 we can derive the exact asymptotics of $C(T_n)$, the cost of the optimal lopsided tree with n leaves, as $n \to \infty$. As described previously this is equivalent to analyzing the costs of Varn codes exactly, solving an open question posed in [19]. The proof of this theorem is given in Section 6.

THEOREM 5. *Let $0 < c_1 \le c_2 \le \cdots \le c_r$ and let T_n be a tree with minimal external path length for those parameters. Let α be the smallest real positive root of the equation $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$ and $\varphi = 1/\alpha$. Let $c = \sum_{i=1}^{r} c_i \varphi^{-c_i}$. Let k and $h = x_k$ be as defined in Lemma 1.*

1. *If $(c_1, \ldots, c_r)$ is rationally related with $\gcd(c_1, \ldots, c_r) = 1$ define*

$$K = \frac{1}{c(1-\varphi^{-1})}\left(\varphi^{\lfloor h\rfloor} - \sum_{i=1}^{k} \varphi^{\lfloor h\rfloor - c_i} + (k-1)\right),$$

$$A = \frac{(k-1)}{c(1-\varphi^{-1})K}, \qquad R = \log_\varphi((1-A)\varphi + A).$$

Then

$$\boxed{C(T_n) = n\log_\varphi n + B\left(\left\{\log_\varphi \frac{n}{K}\right\}\right)n + D\left(\left\{\log_\varphi \frac{n}{K}\right\}\right)n + o(n),}$$

where

$$B(\theta) = h + 1 - \log_\varphi K - \theta - \left(\frac{1}{\varphi - 1} + \{h\}(1-A)\right)\varphi^{1-\theta},$$

$$D(\theta) = \begin{cases} \{h\}(\varphi^{R-\theta} - 1), & h \le c_r \quad \textit{and} \quad \theta \le R, \\ 0, & h \le c_r \quad \textit{and} \quad \theta > R, \\ 0, & h > c_r. \end{cases}$$

If $(\overline{c_1}, \ldots, \overline{c_r}) = d(c_1, \ldots, c_r)$ with $\gcd(c_1, \ldots, c_r) = 1$, then $C(\bar{T}_n) = d \cdot C(T_n)$ where $C(\bar{T}_n)$ is the cost function defined by $(\overline{c_1}, \ldots, \overline{c_r})$.

2. *If $(c_1, \ldots, c_r)$ is irrationally related define $K = (1/(c\ln\varphi))\,(\varphi^h - \sum_{i=1}^{k} \varphi^{h-c_i} + (k-1))$. Then*

$$\boxed{C(T_n) = n\log_\varphi n + \left(h - \log_\varphi K - \frac{1}{\ln\varphi}\right)n + o(n).}$$

See Figures 13 and 14 for examples of how this asymptotic result compares with the real value of $C(T_n)$. These figures also rather nicely illustrate a surprising aspect of the analysis; in the rational case there are sometimes not one but two periodic functions (in $\log_\varphi n$), $B(x)$ and $D(x)$, that arise.

Sometimes, but not always. There are at least two cases in which $D(x) \equiv 0$. The first is when $h > c_r$. This always occurs, for example in the binary case ($r = 2$) when $k = 2$ so $h = c_1 + c_2 > c_2$. The second case is when $\{h\} = 0$, e.g., when h is an integer. Examples of both of these cases are given in Figures 13 and 14. Figure 14 also illustrates how a small change in one of the c_i can cause a switch from $D(x) \equiv 0$ to $D(x) \not\equiv 0$.

4. Derivation of the Structure of Optimal Trees. In this section we derive the proof of Theorem 1.

4.1. *Background*. We start by reviewing what happens in the binary case and describing the difficulty that occurs when trying to extend this analysis to the r-ary tree case.

Recall that optimal trees must be proper. Thus *binary* ($r = 2$) optimal trees for n leaves have $n - 1$ leaves and the procedure described by Theorem 1 creates a sequence of trees T_n^{n-1}, $n = 2, 3, 4, \ldots$. This sequence has the property that T_{n+1}^n is created from T_n^{n-1} by taking the smallest labeled leaf in T_n^{n-1}, making it internal, and adding its two children. Another way of expressing this is that optimal T_{n+1}^n can be created by incrementally *branching*—taking the highest leaf in T_n^{n-1}, making it internal, and adding its two children. This is exactly the procedure described in [19]. See Figure 15 for an example. Note that the $2n - 1$ *nodes* in the optimal tree are not necessarily the $2n - 1$ highest nodes in the infinite tree. By contrast, as we shall soon see, the $n - 1$ *internal nodes* in the optimal tree are the $n - 1$ highest nodes in the infinite tree.

In the r-ary case there are two ways to add a leaf to a tree. The first is again by a *branching* operation which transforms the minimum external node into an internal node, and adds its first two (smallest) children to the tree; the second is by an *adding* operation which adds a previously nonexisting ith child ($2 < i \le r$) of some internal node to the tree. A natural extension of the above incremental algorithm is therefore to consider, at each step, the cost of both *branching* and the minimum cost *adding* operations. If *adding* is cheaper, we perform the *adding*, otherwise we perform the *branching*. (This is the *"extension algorithm"* presented in [26].)

For example, if $r = 5$, $(c_1, c_2, c_3, c_4, c_5) = (3, 3, 3, 8, 8)$, the "extension algorithm" constructs the sequence of trees for 2–6 leaves shown in (a)–(e) of Figure 16, respectively.

However, it is not difficult to see that the tree constructed by the above algorithm is not always optimal. In the above example, the tree for six leaves with cost 34 in (e) is not optimal. The optimal tree T_6 should be (f) of Figure 16 which has cost 32. As pointed out in [26], the algorithm fails because it ignores the fact that it may be worth performing a new *branching* which is more expensive, if it enables a cheaper *adding* later. As in the above example, adding the fifth child of node 1 is cheaper than branching at node 2 but the branching of node 2 enables the cheaper adding of another child (the third child of node 2) later.

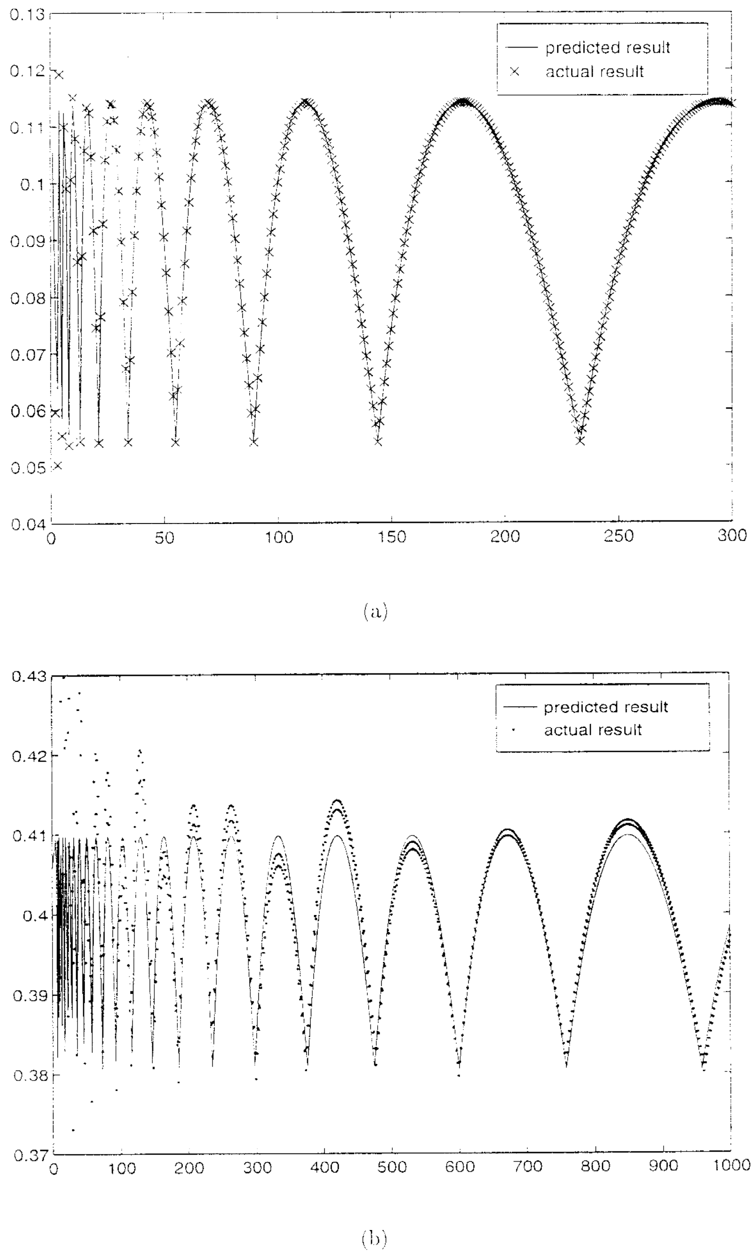
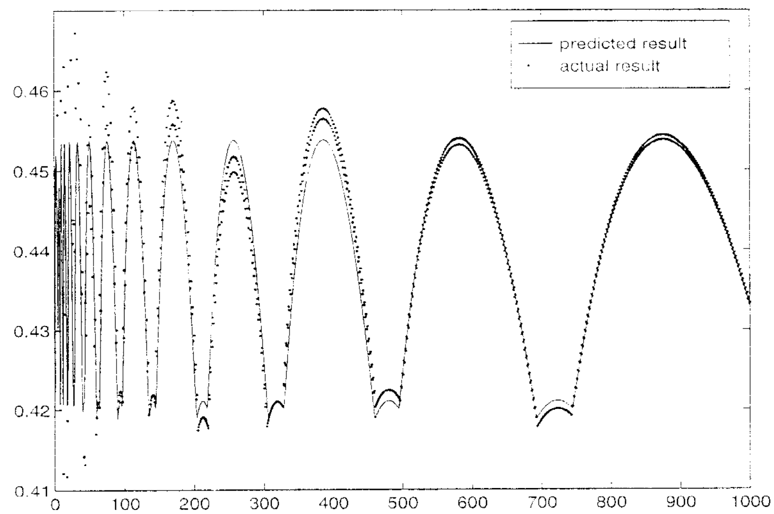
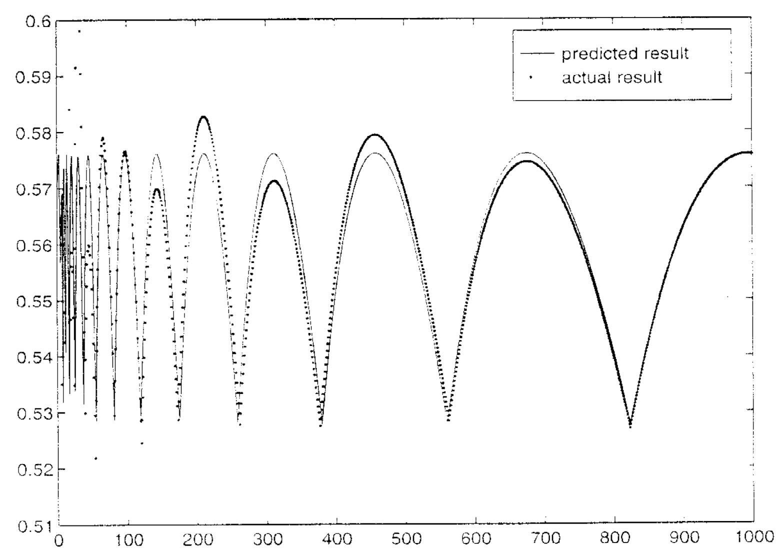


Fig. 13. The predicted cost is $B(\{\log_\varphi(n/K)\}) + D(\{\log_\varphi(n/K)\})$ as defined in Theorem 5, while the actual cost plots function $(1/n)(C(T_n) - n\log_\varphi n)$. (a) $r = 2$ and $(c_1, c_2) = (1, 2)$; $k = 2$, $h = 3$; (b) $r = 3$ and $(c_1, c_2, c_3) = (3, 5, 7)$; $k = 3$, $h = 7.5$. In both cases, since $h > c_r$, $D \equiv 0$.



(a)



(b)

Fig. 14. The predicted cost is $B(\{\log_\varphi(n/K)\}) + D(\{\log_\varphi(n/K)\})$ as defined in Theorem 5, while the actual cost plots function $(1/n)(C(T_n) - n\log_\varphi n)$. (a) $r = 5$ and $(c_1, c_2, c_3, c_4, c_5) = (2, 3, 4, 7, 11)$; $k = 3$, $h = 4.5$; (b) $r = 5$ and $(c_1, c_2, c_3, c_4, c_5) = (2, 3, 5, 7, 11)$; $k = 3$, $h = 5$. In this case, since $\{h\} = 0$, $D \equiv 0$.

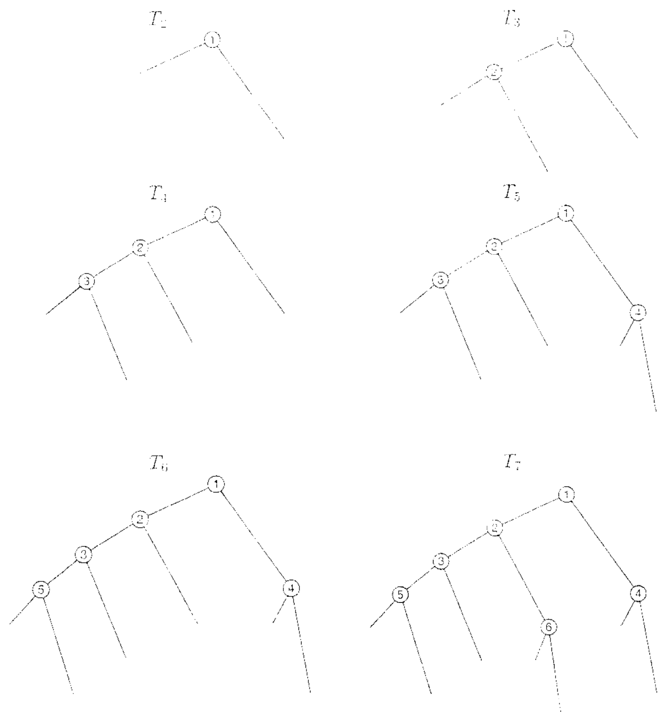


Fig. 15. For $r = 2$, $(c_1, c_2) = (1, 3)$, T_{n+1} is constructed from T_n by making the highest leaf in the tree an internal node with two children.

More precisely, the incremental algorithm fails because in the r-ary case it is not always true that $T_n \subseteq T_{n+1}$, e.g., in the above example $T_5 \not\subseteq T_6$ because the fifth child of node 1 is in T_5 but not in T_6. Perl et al. [26] use a *"mending algorithm"* to change the tree constructed by the *"extension algorithm"* into an optimal one in case it is not already optimal. This algorithm requires $O(nr^2)$ time.

Another approach to constructing optimal trees uses the fact that if m, the number of internal nodes in the optimal tree, is known, then these m nodes can be shown to be the m shallowest (i.e., least-depth) nodes of the infinite tree, while the leaves are the n shallowest available children of these nodes in the infinite tree. This type of tree is called a *shallow tree* in [18]. The trees T_n^m introduced in Definition 6 are, by definition, shallow trees. In the binary case there is a one–one correspondence between the number of external nodes n and the number of internal nodes m, namely, $m = n - 1$. Therefore, in the binary case, an optimal tree T_n for n leaves is exactly the tree containing the highest $(n - 1)$ nodes in the infinite tree as internal nodes, each of which has both of its children in T_n.

However, for the r-ary trees, this kind of one–one correspondence between the number of internal nodes and the number of external nodes does not exist; trees with the same number of external nodes may have different numbers of internal nodes. We do know,

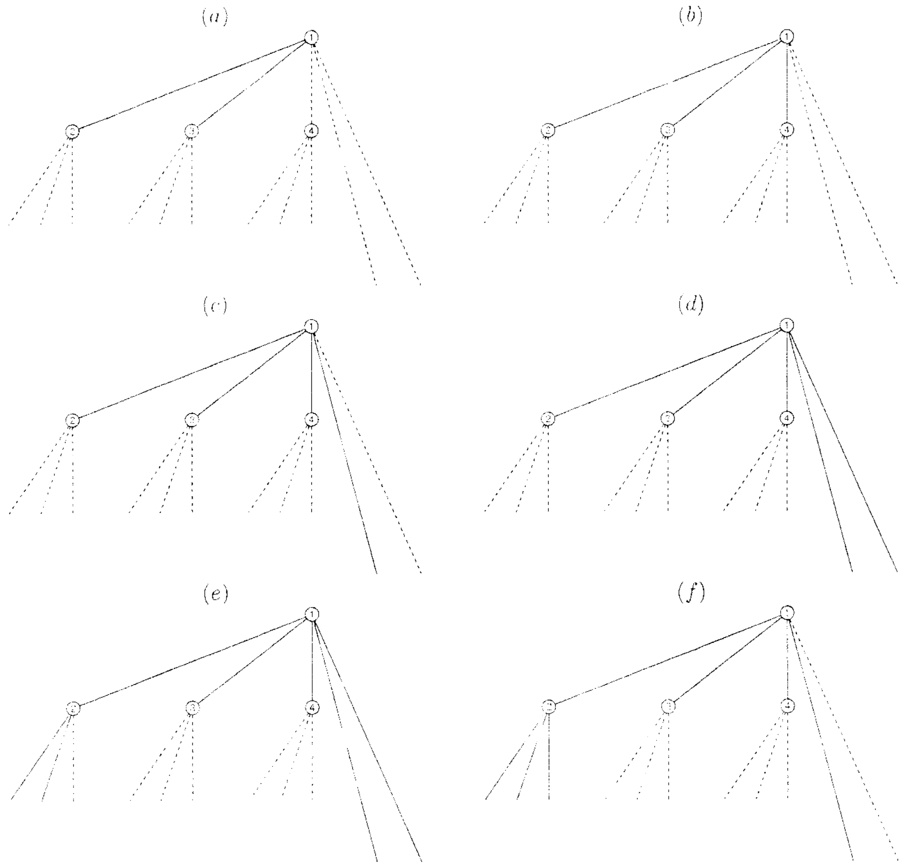


Fig. 16. For $r = 5$, $(c_1, c_2, c_3, c_4, c_5) = (3, 3, 3, 8, 8)$, the extension algorithm constructs the sequence of trees for 2–6 leaves shown in (a)–(e), respectively. The tree in (e) is *not* optimal for six leaves. The tree in (f) is. Notice that T_5 in (d) is not a subtree of T_6 in (f). Dotted lines are infinite tree edges not in the tree.

though, that

LEMMA 3 [18]. *Fix* n *and set* $m_{min} = \lceil (n-1)/(r-1) \rceil \le m \le n-1$. *Let* T_n^m *be as defined in Definition* 6. *Then*:

1. *If* T *is any tree with* n *leaves and* m *internal nodes, then* $C(T_n^m) \le C(T)$.
2. *Let* $m_{\max} = \min\{m\colon T_n^{m+1}$ *is not proper*$\}$. *Then, for all* $m > m_{\max}$, T_n^m *is not proper.*
3. *There exists* m_0, $m_{\min} \le m_0 \le m_{\max}$, *such that* $T_n^{m_0}$ *is optimal*.

This lemma implies that one of the T_n^m must be optimal. To find an optimal tree, it therefore suffices to construct all the T_n^m, $m_{\min} \le m \le m_{\max}$, and return the one with the lowest cost. This, in fact, was the basis for the $O(n \log^2 r)$ time algorithm presented in [18]. See Figure 5 for an example.

In [18] the following observation due to R. Fleischer was reported:

LEMMA 4. *The sequence of tree costs $C(T_n^m)$, $m_{\min} \le m \le m_{\max}$, is convex, i.e., for $m_{\min} < m < m_{\max}$,*

$$(C(T_n^{m+1}) - C(T_n^m)) \ge (C(T_n^m) - C(T_n^{m-1})).$$

In particular, this implies the existence of m_0 such that

$$C(T_n^{m_{\min}}) > \cdots > C(T_n^{m_0-1}) > C(T_n^{m_0}) \le C(T_n^{m_0+1}) \le \cdots \le C(T_n^{m_{\max}}).$$

This lemma, while quite beautiful, did not help at all with the analysis of the algorithm of [18]. In our paper here it will be of tremendous use, though, because it provides a *local* test of the optimality of any particular proper T_n^m. Simply compare T_n^m to its predecessor (if the predecessor exists, i.e., $m > m_{\min}$) and its successor (if the successor is proper, i.e., $m < m_{\max}$). We encapsulate this fact in a lemma:

LEMMA 5. *T_n^m is optimal for n leaves if and only if T_n^m is proper and both of the following are true*:

- *$m = m_{\min}$ or $m > m_{\min}$ and $C(T_n^{m-1}) > C(T_n^m)$.*
- *$m = m_{\max}$ or $m < m_{\max}$ and $C(T_n^{m+1}) \ge C(T_n^m)$.*

4.2. *Evolution of Shallow Trees*. We now try to understand how the cost of T_n^m changes as m increases.

Let n be fixed and let T_n^m and T_n^{m+1} be any two successive proper trees. (Figure 17 shows two successive proper trees.) Recall that $T_n^m = V_m \cup LEAF_n(V_m)$. Rewrite $LEAF_n(V_m) = \{u_1, u_2, \ldots, u_n\}$ where $u_1 < u_2 < \cdots < u_n$ (where we are now using the label of a node as its name).

By definition u_1, the smallest node in $LEAF_n(V_m)$ is the node labeled $m+1$ so $V_{m+1} = V_m \cup \{u_1\}$. Now let $child_i(w)$ denote the ith child of w. Then the set of n smallest children in $LEAF(V_{m+1})$ is the set of n smallest children in

$$\{u_2, \ldots, u_n\} \cup \{child_i(u_1) : i = 1, \ldots, r\}$$

so

$$LEAF_n(V_{m+1}) = \{u_2, \ldots, u_{n+1-d}\} \cup \{child_i(u_1)\colon\ i = 1, \ldots, d\},$$

where

$$\begin{aligned} d &= \max\{i\colon\ child_i(u_1) < u_{n+2-i}, i = 1, \ldots, r\} \\ &= \text{degree of } m+1 \text{ in } T_n^{m+1}. \end{aligned}$$

By assumption, T_n^{m+1} is proper, so $d \ge 2$. Then

$$T_n^{m+1} = V_{m+1} \cup \{u_2, \ldots, u_{n+1-d}\} \cup \{child_i(u_1)\colon\ i = 1, \ldots, d\}.$$

The (long) remainder of this section is devoted to finding d. For later use we define

DEFINITION 7. Let T_n^m be any proper tree. Then $\deg(T_n^m) \stackrel{\text{def}}{=}$ the number of children of internal node m in T_n^m.

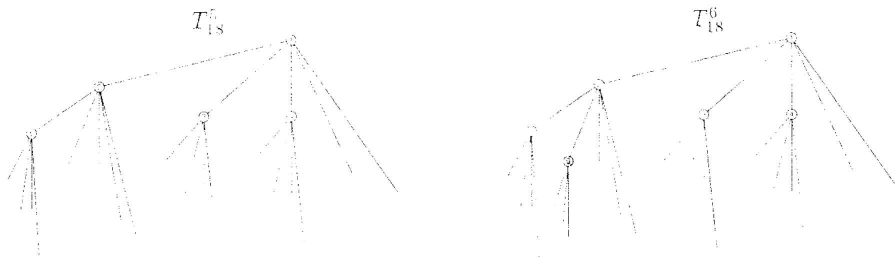


Fig. 17. Two successive trees T_{18}^5 and T_{18}^6. u_1 is node 6 and $d = 3$. The dotted edges are not in the tree.

Note that when m, n are fixed, then $d = \deg(T_n^{m+1})$. Now introduce

DEFINITION 8.

$$\begin{aligned} S(l,t) &\stackrel{\text{def}}{=} \sum_{i=1}^{t}(l+c_i) - l \\ &= \text{cost change in making an external node of depth } l \text{ internal with } t \text{ children}, \\ L(l,t) &\stackrel{\text{def}}{=} t \cdot l \\ &= \text{the cost of } t \text{ nodes of depth } l. \end{aligned}$$

In this notation,

$$\begin{aligned} C(T_n^{m+1}) - C(T_n^m) &= \sum_{i=1}^{d} depth(child_i(u_1)) \\ &\quad - depth(u_1) - \sum_{i=1}^{d-1} depth(u_{n+1-i}) \\ &= S(depth(u_1), d) - \sum_{i=1}^{d-1} depth(u_{n+1-i}). \end{aligned} \tag{12}$$

Note that since u_1 is the highest leaf in T_n^m and is also the deepest internal node in T_n^{m+1},

$$\min_{e \in EX(T_n^m)} depth(e) = depth(u_1) = \max_{v \in IN(T_n^{m+1})} depth(v);$$

since $d \geq 2$ and $u_{n-d+2} \in T_n^{m+1}$,

$$\max_{e \in EX(T_n^m)} depth(e) = depth(u_n) \geq depth(u_{n-d+2}) \geq \min_{u \in UN(T_n^{m+1})} depth(u).$$

Therefore, plugging back into (12),

$$\begin{aligned} &C(T_n^{m+1}) - C(T_n^m) \\ &\quad \leq S\left(\max_{v \in IN(T_n^{m+1})} depth(v), d\right) - L\left(\min_{u \in UN(T_n^{m+1})} depth(u), d-1\right) \end{aligned} \tag{13}$$

and

$$C(T_n^{m+1}) - C(T_n^m) \geq S\left(\min_{e\in EX(T_n^m)} depth(e), d\right) - L\left(\max_{e\in EX(T_n^m)} depth(e), d-1\right), \tag{14}$$

where $d = \deg(T_n^{m+1})$.

The following lemma, previously stated without proof, is crucial in allowing us to take advantage of the previous two equations.

LEMMA 1. *Let $x_m = (\sum_{i=1}^m c_i)/(m-1)$ for $m = 2, \ldots, r$. There exists $k \geq 2$ such that*

$$x_2 \geq x_3 \geq \cdots \geq x_{k-1} \geq x_k < x_{k+1} < \cdots < x_r. \tag{15}$$

(If $x_2 \leq x_3$ set $k = 2$. If $x_2 \geq x_3 \geq \cdots \geq x_{r-1} \geq x_r$ set $k = r$.) Letting k be this value and setting $h \stackrel{\text{def}}{=} x_k$ we have, further, that if $k < r$, then $c_k \leq h < c_{k+1}$.

PROOF. Starting with $x_{m-1} < x_m$,

$$\begin{aligned}
\frac{\sum_{i=1}^{m-1} c_i}{m-2} < \frac{\sum_{i=1}^{m} c_i}{m-1}& \\
\implies\quad & (m-1)\sum_{i=1}^{m-1} c_i < (m-2)\sum_{i=1}^{m} c_i = (m-2)\sum_{i=1}^{m-1} c_i + (m-2)c_m \\
\implies\quad & \sum_{i=1}^{m-1} c_i < (m-2)c_m \\
\implies\quad & \sum_{i=1}^{m} c_i < (m-1)c_m \leq (m-1)c_{m+1} \\
\implies\quad & m\sum_{i=1}^{m} c_i < (m-1)\sum_{i=1}^{m} c_i + (m-1)c_{m+1} = (m-1)\sum_{i=1}^{m+1} c_i \\
\implies\quad & \frac{\sum_{i=1}^{m} c_i}{m-1} < \frac{\sum_{i=1}^{m+1} c_i}{m} \qquad \text{(i.e., } x_m < x_{m+1}\text{)}.
\end{aligned}$$

Therefore, there exists $k \geq 2$ such that

$$x_2 \geq x_3 \geq \cdots \geq x_{k-1} \geq x_k < x_{k+1} < \cdots < x_r,$$

proving (2).

Now let $h \stackrel{\text{def}}{=} x_k$. To prove the second part of the lemma we first use the fact that $x_k \leq x_{k-1}$ so

$$\frac{\sum_{i=1}^{k} c_i}{k-1} \leq \frac{\sum_{i=1}^{k-1} c_i}{k-2} \quad\implies\quad (k-2)\sum_{i=1}^{k} c_i \leq (k-1)\sum_{i=1}^{k-1} c_i$$

$$\Longrightarrow \quad (k-2)c_k \le \sum_{i=1}^{k-1} c_i$$
$$\Longrightarrow \quad (k-1)c_k \le \sum_{i=1}^{k} c_i$$
$$\Longrightarrow \quad c_k \le \frac{\sum_{i=1}^{k} c_i}{k-1} = h.$$

Similarly, since $x_k < x_{k+1}$,

$$\frac{\sum_{i=1}^{k} c_i}{k-1} < \frac{\sum_{i=1}^{k+1} c_i}{k} \quad \Longrightarrow \quad k\sum_{i=1}^{k} c_i < (k-1)\sum_{i=1}^{k+1} c_i$$
$$\Longrightarrow \quad \sum_{i=1}^{k} c_i < (k-1)c_{k+1}$$
$$\Longrightarrow \quad h = \frac{\sum_{i=1}^{k} c_i}{k-1} < c_{k+1}.$$

Thus, $c_k \le h < c_{k+1}$. □

For example, when $r = 5$ and $(c_1, c_2, c_3, c_4, c_5) = (3, 5, 5, 8.75, 10)$ (these are the parameters used in many of our examples), then

$$(x_2, x_3, x_4, x_5) = (8, 6.5, 7.25, 7.9375)$$

so $k = 3$ and $h = 6.5$. Notice that $5 = c_3 \le 6.5 < 8.75 = c_4$.

This last lemma permits bounding the change in the cost of shallow trees as their number of internal nodes grow. We first note that replacing $(d-1)$ leaves of depth $l+h$ by making a leaf of depth l into an internal node with d children will not decrease the cost of the tree.

LEMMA 6. *$S(l, d) \ge L(l+h, d-1)$ for any real l, any integer $d \ge 2$.*

PROOF.

$$\begin{aligned} S(l,d) - L(l+h, d-1) &= \sum_{i=1}^{d}(l + c_i) - l - (d-1)(l+h) \\ &= \sum_{i=1}^{d} c_i - (d-1)h \\ &= (d-1)\left[\frac{\sum_{i=1}^{d} c_i}{d-1} - h\right] \ge 0 \qquad (\text{ since } h \le x_d), \end{aligned}$$

where the last inequality follows from Lemma 1. □

We also note that if an internal node of depth l has exactly k children, then modifying the tree by removing all k children of this node (so that the node itself becomes an external node) and adding $(k-1)$ new leaves with depth larger than $l+h$, will increase the cost of the tree.

LEMMA 7. *For any real* $\delta > 0$, $S(l,k) < L(l+h+\delta, k-1)$. *In particular, if* $l = l_j$ *and* $\delta = l_{j+1} - l_j > 0$, *we have* $S(l_j, k) < L(l_{j+1}+h, k-1)$.

PROOF. Recall that

$$h = \frac{\sum_{i=1}^{k} c_i}{k-1}.$$

Now, for any $\delta > 0$,

$$\sum_{i=1}^{k} c_i = (k-1)h < (k-1)(h+\delta) \quad \Longrightarrow \quad \sum_{i=1}^{k} c_i + (k-1)l < (k-1)(l+h+\delta),$$

i.e., $S(l,k) < L(l+h+\delta, k-1)$. □

We now prove Theorem 1.

PROOF OF THEOREM 1. We prove the theorem case by case. For each case, we prove the optimality of T_n^m by showing both

(a) if $m < m_{\max}$ (i.e., T_n^{m+1} is proper), then $C(T_n^m) \le C(T_n^{m+1})$ and
(b) if $m > m_{\min}$ (i.e., T_n^{m-1} exists), then $C(T_n^{m-1}) > C(T_n^m)$.

The optimality of T_n^m will follow from Lemma 4.

First recall that if T_n^{m+1} is proper, then, by (13),

$$C(T_n^{m+1}) - C(T_n^m) \ \ge\ S\left(\min_{e\in EX(T_n^m)} depth(e), d_1\right) - L\left(\max_{e\in EX(T_n^m)} depth(e), d_1 - 1\right), \tag{16}$$

where $d_1 = \deg(T_n^{m+1}) \ge 2$.

If T_n^{m-1} exists, then, by (14),

$$C(T_n^{m-1}) - C(T_n^m) \ \ge\ L\left(\min_{w\in UN(T_n^m)} depth(w), d_2 - 1\right) - S\left(\max_{v\in IN(T_n^m)} depth(v), d_2\right), \tag{17}$$

where $d_2 = \deg(T_n^m)$.

1. $n = a_j$, $T_{a_j}^{m_j} = V_{m_j} \cup A_j$. (See Figure 8.) Clearly, $T_{a_j}^{m_j}$ is proper and

$$\min_{e\in EX(T_{a_j}^{m_j})} depth(e) \ =\ l_{j+1}, \qquad \max_{e\in EX(T_{a_j}^{m_j})} depth(e) \le l_j + h,$$

$$\max_{v\in IN(T_{a_j}^{m_j})} depth(v) \ =\ l_j,$$

and $\deg(T_{a_j}^{m_j}) = k$ since $c_k \le h < c_{k+1}$ (Lemma 1).

Hence, if $T_{a_j}^{m_j+1}$ is proper, then, by (16),

$$\begin{aligned} C(T_{a_j}^{m_j+1}) - C(T_{a_j}^{m_j}) &\geq S(l_{j+1}, d_1) - L(l_j + h, d_1 - 1) \\ &\geq S(l_j, d_1) - L(l_j + h, d_1 - 1) \\ &\geq 0 \qquad \text{(by Lemma 6)}, \end{aligned}$$

proving (a).

If $T_{a_j}^{m_j-1}$ exists, then

$$\min_{w \in UN(T_{a_j}^{m_j})} depth(w) = l_j + h + \delta, \qquad \text{for some} \quad \delta > 0,$$

and, by (17),

$$\begin{aligned} C(T_{a_j}^{m_j-1}) - C(T_{a_j}^{m_j}) &\geq L(l_j + h + \delta, k - 1) - S(l_j, k) \\ &> 0 \qquad \text{(by Lemma 7)}, \end{aligned}$$

proving (b).

Therefore, $T_n = T_{a_j}^{m_j}$ is optimal for n leaves.

2. $a_j < n \leq b_j$, $T_n^{m_j} = V_{m_j} \cup LEAF_n(V_{m_j})$. (See Figure 9.) Clearly, $T_n^{m_j}$ is proper and

$$\min_{e \in EX(T_n^{m_j})} depth(e) = l_{j+1}, \qquad \max_{e \in EX(T_n^{m_j})} depth(e) \leq l_{j+1} + h,$$
$$\max_{v \in IN(T_n^{m_j})} depth(v) = l_j.$$

Hence, if $T_n^{m_j+1}$ is proper, then, by (16),

$$\begin{aligned} C(T_n^{m_j+1}) - C(T_n^{m_j}) &\geq S(l_{j+1}, d_1) - L(l_{j+1} + h, d_1 - 1) \\ &\geq 0 \qquad \text{(by Lemma 6)}, \end{aligned}$$

proving (a).

If $T_n^{m_j-1}$ exists, then

$$\min_{w \in UN(T_n^{m_j})} depth(w) = l_j + h + \delta, \qquad \text{for some} \quad \delta > 0$$

and $d = \deg(T_n^{m_j}) \geq k$ since $h \geq c_k$ (Lemma 1). By (17),

$$\begin{aligned} C(T_n^{m_j-1}) - C(T_n^{m_j}) &\geq L(l_j + h + \delta, d - 1) - S(l_j, d) \\ &= L(l_j + h + \delta, k - 1) + (d - k)(l_j + h + \delta) \\ &\quad - S(l_j, k) - (l_j + c_{k+1}) - \cdots - (l_j + c_d). \end{aligned}$$

Since $l_j + c_i \leq \min_{w \in UN(T_n^{m_j})} depth(w) = l_j + h + \delta$, for $i \leq d$,

$$\begin{aligned} C(T_n^{m_j-1}) - C(T_n^{m_j}) &\geq L(l_j + h + \delta, k - 1) - S(l_j, k) \\ &> 0 \qquad \text{(by Lemma 7)}, \end{aligned}$$

proving (b).

Therefore, $T_n = T_n^{m_j}$ is optimal for n leaves.

3. $b_j < n \le a_{j+1}$. We write $n = b_j + p(k-1) + q$, where $p = \lfloor (n - b_j)/(k-1) \rfloor$ and $q = (n - b_j) \bmod (k-1)$. Before starting, let L_j be the set of nodes on level l_j. From the definitions of a_j and b_j it is not hard to see that $a_{j+1} = b_j + (k-1)|L_{j+1}|$.

Thus, for n in the range that we are examining, $p \le |L_j|$ with $p = L_j$ if and only if $n = a_{j+1}$ (in which case $q = 0$). Note that if $n = a_{j+1}$, then the tree constructed by this part of the theorem is exactly $T_{a_{j+1}}^{m_{j+1}}$ (see the comments following the statement of Theorem 1 for a more detailed explanation) which, by the first part of the theorem, is already known to be optimal. For this reason we restrict the remainder of the proof to $n < a_{j+1}$ which in turn implies that $p < |L_{j+1}|$.

There are two cases:

(A) $p > 0$ *and* $q = 0$. $T_n^{m_j+p} = V_{m_j+p} \cup LEAF_n(V_{m_j+p})$. (See Figure 10.) Clearly, $T_n^{m_j+p}$ is proper. Since $p < |L_{j+1}|$ at least one node in L_{j+1} remains in $EX(T_n^{m_j+p})$ implying that

$$\min_{e \in EX(T_n^{m_j+p})} depth(e) = l_{j+1}.$$

As in the previous parts of the proof,

$$\max_{e \in EX(T_n^{m_j+p})} depth(e) \le l_{j+1} + h, \qquad \max_{v \in IN(T_n^{m_j+p})} depth(v) = l_{j+1},$$

and $\deg(T_n^{m_j+p}) = k$ since $c_k \le h < c_{k+1}$.

Hence, if $T_n^{m_j+p+1}$ is proper, then, by (16),

$$\begin{aligned} C(T_n^{m_j+p+1}) - C(T_n^{m_j+p}) &\ge S(l_{j+1}, d_1) - L(l_{j+1} + h, d_1 - 1) \\ &\ge 0 \qquad \text{(by Lemma 6)}, \end{aligned}$$

proving (a).

If $T_n^{m_j+p-1}$ exists, then

$$\min_{w \in UN(T_n^{m_j+p})} depth(w) = l_{j+1} + h + \delta, \qquad \text{for some} \quad \delta > 0$$

and, by (17),

$$\begin{aligned} C(T_n^{m_j+p-1}) - C(T_n^{m_j+p}) &\ge L(l_{j+1} + h + \delta, k-1) - S(l_{j+1}, k) \\ &> 0 \qquad \text{(by Lemma 7)}, \end{aligned}$$

proving (b).

Therefore, $T_n = T_n^{m_j+p}$ is optimal for n leaves.

(B) $p \ge 0$ *and* $k - 1 > q > 0$. Note that if $m_j + p > m_{\min} = \lceil (n-1)/(r-1) \rceil$, then $T_n^{m_j+p}$ exists. (See Figure 11.)

Case (i): *Both* $T_n^{m_j+p}$ *exists and* $C(T_n^{m_j+p+1}) \ge C(T_n^{m_j+p})$. In this case we prove $T_n = T_n^{m_j+p}$ is optimal. To do so we must show that $C(T_n^{m_j+p-1}) > C(T_n^{m_j+p})$ if $T_n^{m_j+p-1}$ exists.

Suppose that $T_n^{m_j+p-1}$ exists. Then

$$\min_{w \in UN(T_n^{m_j+p})} depth(w) = l_{j+1} + h + \delta, \qquad \text{for some} \quad \delta > 0.$$

If $p = 0$, then

$$\max_{v \in IN(T_n^{m_j+p})} depth(v) = l_j$$

and $d = \deg(T_n^{m_j}) \geq k$ since $h \geq c_k$ (Lemma 1). By (17),

$$\begin{aligned} C(T_n^{m_j+p-1}) - C(T_n^{m_j+p}) &\geq L(l_{j+1} + h + \delta, d - 1) - S(l_j, d) \\ &= L(l_{j+1} + h + \delta, k - 1) + (d - k)(l_{j+1} + h + \delta) \\ &\quad - S(l_j, k) - (l_j + c_{k+1}) - \cdots - (l_j + c_d). \end{aligned}$$

Since $l_j + c_i \leq \min_{w \in UN(T_n^{m_j+p})} depth(w) = l_{j+1} + h + \delta$, for $i \leq d$,

$$\begin{aligned} C(T_n^{m_j+p-1}) - C(T_n^{m_j+p}) &\geq L(l_{j+1} + h + \delta, k - 1) - S(l_j, k) \\ &> L(l_j + h + \delta, k - 1) - S(l_j, k) \\ &> 0 \qquad \text{(by Lemma 7)}, \end{aligned}$$

proving (b).

Otherwise $p > 0$,

$$\max_{v \in IN(T_n^{m_j+p})} depth(v) = l_{j+1}$$

and $\deg(T_n^{m_j+p}) = k$. By (17),

$$\begin{aligned} C(T_n^{m_j+p-1}) - C(T_n^{m_j+p}) &\geq L(l_{j+1} + h + \delta, k - 1) - S(l_{j+1}, k) \\ &> 0 \qquad \text{(by Lemma 7)}, \end{aligned}$$

proving (b).

Therefore, $T_n = T_n^{m_j+p}$ is optimal for n leaves.

Case (ii): *Either* $T_n^{m_j+p}$ *does not exist or* $C(T_n^{m_j+p+1}) < C(T_n^{m_j+p})$. In this case we prove that $T_n^{m_j+p+1}$ is optimal. To do so we must show that $C(T_n^{m_j+p+2}) \geq C(T_n^{m_j+p+1})$ if $T_n^{m_j+p+2}$ is proper.

$$\min_{e \in EX(T_n^{m_j+p+1})} depth(e) \geq l_{j+1}, \qquad \max_{e \in EX(T_n^{m_j+p+1})} depth(e) \leq l_{j+1} + h.$$

Hence, if $T_n^{m_j+p+2}$ is proper, then, by (16),

$$\begin{aligned} C(T_n^{m_j+p+2}) - C(T_n^{m_j+p+1}) &\geq S(l_{j+1}, d_1) - L(l_{j+1} + h, d_1 - 1) \\ &\geq 0 \qquad \text{(by Lemma 6)}, \end{aligned}$$

proving (a).

Therefore, $T_n = T_n^{m_j+p+1}$ is optimal for n leaves. □

5. Analysis of *F(x)*. In this section we prove Theorem 2. Recall its statement: *Let* $(c_1, \ldots, c_r)$ *be an r-tuple of nonnegative reals and define*

$$F(x) = \begin{cases} 1 + F(x-c_1) + F(x-c_2) \\ \qquad + \cdots + F(x-c_r), & \text{if} \quad x \geq c_1; \\ 1, & \text{if} \quad 0 \leq x < c_1; \\ 0, & \text{if} \quad x < 0. \end{cases} \tag{18}$$

Let α *be the smallest real positive root of the equation* $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$ *and* $\varphi = 1/\alpha$. *Let* $c = (\sum_{i=1}^{r} c_i \varphi^{-c_i})$. *Then*:

1. *If* $(c_1, \ldots, c_r)$ *is rationally related,*

$$F(x) = D(x)\varphi^x + o(\rho^x),$$

   where $D(x) = (d/c(1-\varphi^{-d}))\varphi^{-d\{x/d\}}$ *is a periodic function with period d and* $0 \leq \rho < \varphi$.
2. *If* $(c_1, \ldots, c_r)$ *is irrationally related,*

$$F(x) = \frac{1}{c \ln \varphi}\varphi^x + o(\varphi^x).$$

PROOF. Our proof proceeds in stages. We first restrict the analysis to the case in which the $(c_1, \ldots, c_r)$ are integers such that $\gcd(c_1, \ldots, c_r) = 1$ and prove the correctness of the theorem using generating functions. We then show how to scale this result to prove the theorem for all rationally related cases. We conclude by analyzing the irrationally related case via Mellin transform-like techniques. To start we need the following simple lemma:[6]

LEMMA 8. *Let* $Q(z) = 1 - z^{c_1} - z^{c_2} - \cdots - z^{c_r}$ *where either the* $(c_1, \ldots, c_r)$ *are positive integers such that* $\gcd(c_1, \ldots, c_r) = 1$ *or the* $(c_1, \ldots, c_r)$ *are irrationally related. Let* α *be the smallest positive root of* $Q(z)$. *Then*:

1. α *is a simple root with* $0 < \alpha < 1$.
2. *If* $Q(z) = 0$ *and* $z \neq \alpha$, *then* $|z| > \alpha$.

PROOF. Notice first that $\alpha \in (0, 1)$ because $1 = Q(0) > 0 > Q(1) = 1 - r$. Furthermore, $Q'(\alpha) \neq 0$, so α must be a simple root.

To prove part 2 suppose that $z = \beta e^{i\theta}$ is another root with $0 \leq \beta \leq \alpha$, $0 \leq \theta < 2\pi$. If $\beta \leq \alpha$, then

$$\begin{aligned} \Re(Q(z)) &= 1 - \beta^{c_1}\Re(e^{ic_1\theta}) - \beta^{c_2}\Re(e^{ic_2\theta}) - \cdots - \beta^{c_r}\Re(e^{ic_r\theta}) \\ &\geq 1 - \alpha^{c_1} - \alpha^{c_2} - \cdots - \alpha^{c_r} = 0, \end{aligned}$$

where $\Re(z)$ is the real part of z. Equality holds in the equation if and only if $\beta = \alpha$ and $\Re(e^{ic_j\theta}) = 1$ for all $1 \leq j \leq r$. In other words there must exist positive integers k_j such

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[6] The authors thank Xavier Gourdon for suggesting this approach.

that $c_j\theta = 2\pi k_j$, $1 \le j \le r$. This in turn implies $c_j = (2\pi/\theta)k_j$ contradicting either the fact that $\gcd(c_1, \ldots, c_r) = 1$ in the rational case or that the $(c_1, \ldots, c_r)$ are irrationally related. Therefore $Q(z) = 0$ and $|z| \le \alpha$ imply $z = \alpha$. □

We can now prove the theorem for the case that the $(c_1, \ldots, c_r)$ are integers with $\gcd(c_1, \ldots, c_r) = 1$ through the use of straightforward generating-function techniques. Referring back to (3) we see that

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} F(n) z^n \\ &= \sum_{n=0}^{c_1-1} z^n + \sum_{n=c_1}^{\infty} [1 + F(n-c_1) + F(n-c_2) + \cdots + F(n-c_r)] z^n \\ &= \sum_{n=0}^{c_1-1} z^n + \sum_{n=c_1}^{\infty} z^n + \sum_{n=0}^{\infty} F(n-c_1) z^n + \cdots + \sum_{n=0}^{\infty} F(n-c_r) z^n \\ &= \frac{1}{1-z} + z^{c_1} \sum_{n=0}^{\infty} F(n) z^n + \cdots + z^{c_r} \sum_{n=0}^{\infty} F(n) z^n \\ &= \frac{1}{1-z} + (z^{c_1} + \cdots + z^{c_r}) G(z) \end{aligned}$$

so

$$G(z) = \frac{1}{(1-z)(1-z^{c_1} - \ldots - z^{c_r})} = \frac{1}{(1-z)Q(z)}.$$

Now, $F(n)$ is the coefficient of z^n in $G(z)$ so, by Lemma 8 and standard generating-function techniques [14], $F(n) = k\varphi^n + O(\rho^n)$, where α is the smallest root of $Q(z)$, $\varphi = 1/\alpha$,

$$k = -\frac{1}{\alpha(1-\alpha)(dQ/d\alpha)(\alpha)} = \frac{1}{(1-\varphi^{-1})} \frac{1}{\sum_i c_i \varphi^{-c_i}} = \frac{1}{c(1-\varphi^{-1})},$$

and $0 < \rho < \varphi$ (actually ρ can be taken to be any value $1/\alpha' - \varepsilon$ where α' is the modulus of the second smallest modulus root of $Q(z)$ and $\varepsilon \ge 0$ is any arbitrary value). Since $F(x)$ only changes at integral values of x this requires

$$F(x) = F(\lfloor x \rfloor) = \frac{c}{1-\varphi^{-1}} \varphi^{-\{x\}} \varphi^x + O(\rho^x)$$

proving the theorem in the rational case when $\gcd(c_1, \ldots, c_r) = 1$.

To prove the theorem for the rational case in which $d = \gcd(c_1, \ldots, c_r) \ne 1$ let $({c_1}', \ldots, {c_r}') = (1/d)(c_1, \ldots, c_r)$. Then $\gcd({c_1}', \ldots, {c_r}') = 1$. Define

$$F'(x) = \begin{cases} 1 + F'(x - {c_1}') + F'(x - {c_2}') \\ \qquad + \cdots + F'(x - {c_r}'), & \text{if } x \ge {c_1}'; \\ 1, & \text{if } 0 \le x < {c_1}'; \\ 0, & \text{if } x < 0; \end{cases} \tag{19}$$

set $Q'(z) = 1 - z^{c'_1} - z^{c'_2} - \cdots - z^{c'_r}$, let α' be its smallest positive root and $\varphi' = 1/\alpha'$.

From (6) it is easy to see that, for all $x \geq 0$, $F(x) = F'(x/d)$. Furthermore, $Q(x) = Q'(x^d)$ so $\varphi' = \varphi^d$. Finally

$$c' = \sum_i c_i{}'(\varphi')^{-c_i{}'} = \frac{1}{d}\sum_i c_i \varphi^{-c_i} = \frac{c}{d}.$$

Combining all of these facts yields

$$\begin{aligned} F(x) \;&=\; F'\left(\frac{x}{d}\right) = \frac{1'}{c'(1-\varphi'^{-1})}\varphi'^{-\{x/d\}}\varphi'^{x/d} + O((\rho')^{x/d}) \\ &=\; \frac{d}{c(1-\varphi^{-d})}\varphi^{-d\{x/d\}}\varphi^{x} + O(\rho^{x}), \end{aligned}$$

where $\rho' < \varphi'$ and $\rho = (\rho')^{1/d} < (\varphi')^{1/d} = \varphi$. This proves the theorem for all rational cases. To prove it for the irrational case we calculate the Mellin transform of $F(\ln x)$ and find the asymptotics of $F(\ln x)$ by taking the inverse Mellin transform. To perform this last step we use the following lemma due to Fredman and Knuth [15] which is in turn a modification of an earlier result due to Landau. In this lemma $f(x) \sim g(x)$ denotes that $f(x) = g(x) + o(g(x))$:

LEMMA 9 [15, Lemma 4.3] . *Let $f(t)$ be a nondecreasing function of the real variable t, with $f(t) \geq 0$. Assume that $G(s) = \int_1^\infty f(t)\,dt/t^{s+1}$ is an analytic function of the complex variable s when $\Re(s) \geq \gamma > 0$, except for a first-order pole at $s = \gamma$ with positive residue C. Then $f(t) \sim Ct^\gamma$.*

Note: If, as will occur in the functions $f(t)$ that we examine, $f(t) = 0$ for $t \leq 1$, then $G(-s)$ is the Mellin transform of $f(t)$ and the lemma is revealed to be a special case of the inversion theorem for Mellin transforms.[7]

Define the function $f(t)$ by $f(t) = 0$ for $t \leq 1$ and $f(t) = F(\ln t)$ for $t \geq 1$. Setting $d_j = e_j^c$ we find that, for $t \geq 1$,

$$\begin{aligned} f(t) \;&=\; 1 + F(\ln t - c_1) + F(\ln t - c_2) + \cdots + F(\ln t - c_r) \\ &=\; 1 + f\left(\frac{t}{d_1}\right) + f\left(\frac{t}{d_2}\right) + \cdots + f\left(\frac{t}{d_r}\right). \end{aligned}$$

We now show that $G(s) = \int_1^\infty f(t)\,dt/t^{s+1}$ satisfies the conditions of the lemma; we can therefore apply it to find the asymptotics of $f(t)$ and ultimately $F(x)$.

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[7] We are indebted to one of the anonymous referees for pointing out an improvement to our proof. With the exception of the pole at $s = \gamma$, the function $G(s)$ can be analytically continued into the halfplane $\{s\colon \Re(s) > \gamma - \delta\}$ for some $\delta > 0$. It is therefore possible to derive the proof of the theorem in the irrational case directly by using the Mellin transform inversion formula as described in [13] instead of employing Lemma 5. This alternative proof would then improve our result to show that $F(x) = (1/(c\ln\varphi))\varphi^x + O(\rho^x)$ for some $0 \leq \rho < \varphi$ instead of just $F(x) = (1/(c\ln\varphi))\varphi^x + o(\varphi^x)$.

Set α and φ as in the theorem statement and define $\gamma = \ln \varphi$. Let $(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_r})$ be any tuple such that $\overline{c_i} \le c_i$ for all $1 \le i \le r$, and let $\bar{F}(x)$ be the "number of nodes" function associated with $(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_r})$. The characterization of $F(x)$ given by (6) shows that $F(x) \le \bar{F}(x)$.

Now suppose that $(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_r})$ are rationally related and let $\bar{\varphi}$ be the reciprocal of the smallest positive root of $1 - z^{\overline{c_1}} - z^{\overline{c_2}} - \cdots - z^{\overline{c_r}}$. Then $F(x) \le \bar{F}(x) = \theta(\bar{\varphi}^x)$. As $(\overline{c_1}, \overline{c_2}, \ldots, \overline{c_r})$ approaches closer and closer to $(c_1, \ldots, c_r)$ continuity implies that $\bar{\varphi} \to \varphi$, implying that $F(x) = O((\varphi + \varepsilon)^x)$ for every $\varepsilon > 0$. Thus, for every fixed $\varepsilon' > 0$,

$$f(t) = F(\ln t) = O((\varphi + \varepsilon)^{\ln t}) = O(t^{\gamma + \varepsilon'})$$

(where the constant in the $O()$ might depend upon ε'). This in turn proves that $G(s)$ converges uniformly and is analytic in the halfplane $\{s | \Re(s) > \gamma + \varepsilon'\}$. We can therefore solve for $G(s)$ in that halfplane as follows:

$$\begin{aligned} G(s) &= \int_1^\infty f(t)\, \frac{dt}{t^{s+1}} \\ &= \int_1^\infty \frac{dt}{t^{s+1}} + \int_1^\infty f\left(\frac{t}{d_1}\right) \frac{dt}{t^{s+1}} + \cdots + \int_1^\infty f\left(\frac{t}{d_r}\right) \frac{dt}{t^{s+1}} \\ &= \frac{1}{s} + [d_1^{-s} + d_2^{-s} + \cdots + d_r^{-s}] G(s) \end{aligned}$$

from which we derive

$$G(s) = \frac{1}{s(1 - d_1^{-s} + d_2^{-s} + \cdots + d_r^{-s})}. \tag{20}$$

This equation is valid in the halfplane $\{s | \Re(s) > \gamma\}$. Now notice that

$$1 - d_1^{-s} + d_2^{-s} + \cdots + d_r^{-s} = 1 - e^{-c_1 s} - e^{-c_2 s} + \cdots + e^{-c_r s} = Q(e^{-s}).$$

Therefore $\gamma = \ln \varphi = -\ln \alpha$ is a pole of $G(s)$ and from Lemma 8 all poles of $G(s)$ must be on or to the left of the line $\{s | \Re(s) = \gamma\}$. Furthermore, $\ln \varphi$ is the only pole on that line. This is because for $s = \varphi + iy$, $y \ne 0$, we have

$$\begin{aligned} \Re(1 - e^{-c_1 s} - \cdots - e^{-c_r s}) &= 1 - \alpha^{c_1} \Re(e^{-c_1 iy}) - \cdots - \alpha^{c_r} \Re(e^{-c_r yi}) \\ &\ge 1 - \alpha^{c_1} - \cdots - \alpha^{c_r} = 0 \end{aligned}$$

with equality if and only if $\Re(e^{-c_j yi}) = 1$ for all j, i.e., there exist positive integers k_j such that $c_j y = 2\pi k_j$ or $c_j = 2\pi k_j / y$ contradicting the fact that $(c_1, \ldots, c_r)$ are irrationally related.

We can therefore analytically continue $G(s)$ over and to the left of the line $\{s | \Re(s) = \gamma\}$ and the analytic continuation has a first-order pole at $s = \gamma$ but no other singularity on that line. The residue of $G(s)$ at $s = \gamma$ is

$$\frac{1}{\gamma \sum_i \ln d^i\, d_i^{-\gamma}} = \frac{1}{\ln \varphi \sum_i c_i \varphi^{-c_i}}.$$

Applying Lemma 9 we find

$$F(x) = f(e^x) = \frac{1}{\gamma \sum_i \varphi^{-c_i} c_i} e^{\gamma x} + o(e^{\gamma x}) = \frac{1}{\ln \varphi \, c} \varphi^x + o(\varphi^x)$$

proving the theorem.

Note: We mention here that it is actually possible to use Mellin transform techniques to analyze the rational case as well. The reason we do not do so is that in the rational case the line $\{s | \Re(s) = \gamma\}$ will contain an infinite number of poles, all of whose residues must be added together (yielding a Fourier series representation of the periodic function); the generating function technique yields a simpler representation of the answer. □

6. The Cost of Optimal Trees. In this section we combine our knowledge of the combinatorial structure of optimal trees T_n (Theorem 1) with our analysis of $F(x)$ (Theorem 2) to derive the proof of Theorem 5 describing how the costs of Varn codes grow as n increases.

We divide the proof into three parts, each of which has its own subsection. In the first part we derive some general lemmas, true for *all* choices of $(c_1, \ldots, c_r)$, describing the growth of costs of optimal trees. In the second part we specialize this lemma to the irrational $(c_1, \ldots, c_r)$. We conclude in the third part by specializing the lemmas to rational $(c_1, \ldots, c_r)$. The analyses of the rational and irrational parts, taken together, prove the theorem.

As we will soon see, the analysis of the rational case is much more technically complicated than that of the irrational one, The intuitive reason for the difference in difficulties is that our approach is to calculate $C(T_n)$ by first finding j such that $a_j \le n < a_{j+1}$, calculating $C(T_{a_j})$ and $C(T_n) - C(T_{a_j})$, and then combining them to get $C(T_n)$. In the irrational case, as j increases, the optimal trees grow smoothly (this is the content of Lemma 12), i.e., the difference between successive depths in the tree tend to zero and the number of nodes per any *individual* level will be relatively small (since the nodes are distributed among many levels). This relative paucity of nodes on the bottom level implies that $C(T_n) - C(T_{a_j}) = o(C(T_n))$, and we will therefore be able to approximate $C(T_n)$ by $C(T_{a_j})$, which is much easier to calculate. In the rational case the levels in the tree are equally spaced and, more importantly, the number of nodes on successive levels grow geometrically. This implies that the number of nodes on the bottom level will always be a constant fraction of the total nodes in the tree. This fact requires that our analysis of the contribution of nodes on the bottom level must be delicate since $C(T_n) - C(T_{a_j})$ could be a substantial fraction of $C(T_n)$. It is this which leads to much of the complications.

6.1. *General Cost Lemmas.* Recall that T_{a_j} is the optimal tree which has as internal nodes the set V_{m_j} consisting of all nodes in the infinite tree at depth l_j or above. In Section 3.1 we also defined

$$A_j = \{v \in LEAF(V_{m_j}) \colon depth(v) \le l_j + h\},$$

$$B_j = A_j \cup \{v \in LEAF(V_{m_j}) \colon l_j + h < depth(v) \le l_{j+1} + h\},$$

$a_j = |A_j|$, and $b_j = |B_j|$. A_j is the set of leaves in T_{a_j}. The highest nodes in A_j are the $m_{j+1} - m_j$ nodes at depth l_{j+1} in the tree. Since node labeling is consistent with the depth ordering, the nodes on depth l_{j+1} are labeled

$$m_j + 1,\ m_j + 2,\ m_j + 3,\ \ldots,\ m_j + l_{j+1}.$$

Finally let

$$u_1 < u_2 < u_3 < \cdots < u_{b_j - a_j}$$

be the labels of the nodes in $B_j \backslash A_j$.

With these definitions we can now prove:

LEMMA 10. *Given n let j be such that* $a_j \le n < a_{j+1}$. *There are three possible cases*:

- *If* $a_j \le n \le b_j$, *then*

$$C(T_n) = C(T_{a_j}) + \sum_{i=1}^{n-a_j} depth(u_i). \tag{21}$$

In particular,

$$C(T_{b_n}) = C(T_{a_j}) + \sum_{i=1}^{b_j-a_j} depth(u_i). \tag{22}$$

- *If* $n = b_j + p(k-1)$, *then*

$$C(T_n) = C(T_{b_j}) + p(k-1)(l_{j+1} + h). \tag{23}$$

- *If* $n = b_j + p(k-1) + q$ *with* $0 < q < k$, *then*

$$C(T_n) = C(T_{b_j}) + (p(k-1) + q)(l_{j+1} + h) + O(1). \tag{24}$$

PROOF. The first part follows directly from Theorem 1, part 2.

To prove the second part note that if $n = b_j + p(k-1)$, then from Theorem 1, part 3, T_n is created by starting with T_{b_j}, taking p of its leaves from depth l_{j+1} and making each of them internal with k leaves. The change in cost resulting from making one such leaf internal is

$$-l_{j+1} + \sum_{i=1}^{k} (l_{j+1} + c_i) = (k-1)(l_{j+1} + h),$$

where we are using the definition $h = (\sum_{i=1}^{k} c_i)/(k-1)$. The proof of the second part follows.

To prove the third part we recall from the discussion following the statement of Theorem 1 that $T_{b_j+p(k-1)+q}$ is created either by starting with $T^{m_j+p}_{b_j+p(k-1)}$ and adding the q smallest unused leaves in V_{m_j+p} to the tree or starting with $T^{m_j+p+1}_{b_j+(p+1)(k-1)}$ and erasing the $k-1-q$ deepest leaves in that tree. In both cases the nodes added or subtracted must have depth at least l_{j+1} and at most $l_{j+1} + c_r$ (since all of their parents have depth most l_{j+1}). Thus, in both cases the third part follows from (23). □

We now observe that $F(x)$ is a step function which jumps $(F(l_i) - F(l_{i-1}))$ at l_i. For $\delta > 0$, we may thus express $F(l_j + \delta) - F(l_j)$ by the Riemann–Stieltjes integral,

$$F(l_j + \delta) - F(l_j) = \int_{l_j}^{l_j+\delta} dF(x).$$

This notation makes our analysis somewhat easier.

We start by deriving an expression for $Cost(T_{a_j})$ that we afterwards combine with Lemma 10 to yield an expression for general T_n.

LEMMA 11.

$$a_j = \int_{l_j}^{l_j+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} dF(x), \tag{25}$$

$$b_j = \int_{l_j}^{l_{j+1}+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_{j+1}+h-c_i} dF(x), \tag{26}$$

$$C(T_{a_j}) = (l_j + h)a_j - \int_{l_j}^{l_j+h} F(x)\,dx + \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} F(x)\,dx. \tag{27}$$

PROOF. By definition,

$$\begin{aligned}
A_j &= \{v \in LEAF(V_{m_j})\colon\ depth(v) \le l_j + h\} \\
&= \bigcup_{i=1}^{r} \{child_i(u)\colon\ u \in V_{m_j} \text{ and } l_j < depth(child_i(u)) \le l_j + h\} \\
&= \{v\colon\ l_j < depth(v) \le l_j + h\} \setminus \bigcup_{i=1}^{r} \{child_i(u)\colon\ l_j < depth(u) \le l_j + h - c_i\} \\
&= \{v\colon\ l_j < depth(v) \le l_j + h\} \setminus \bigcup_{i=1}^{k} \{child_i(u)\colon\ l_j < depth(u) \le l_j + h - c_i\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_j &= |\{v\colon\ l_j < depth(v) \le l_j + h\}| - \sum_{i=1}^{k} |\{child_i(u)\colon\ l_j < depth(u) \le l_j + h - c_i\}| \\
&= |\{v\colon\ l_j < depth(v) \le l_j + h\}| - \sum_{i=1}^{k} |\{u\colon\ l_j < depth(u) \le l_j + h - c_i\}| \\
&= \int_{l_j}^{l_j+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} dF(x).
\end{aligned}$$

Similarly,

$$b_j = \int_{l_j}^{l_{j+1}+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_{j+1}+h-c_i} dF(x)$$

and

$$\begin{aligned} C(T_{a_j}) &= \sum_{v \in A_j} depth(v) \\ &= \sum_{l_j < depth(v) \le l_j + h} depth(v) - \sum_{i=1}^{k} \sum_{l_j < depth(u) \le l_j + h - c_i} depth(child_i(u)) \\ &= \sum_{l_j < depth(v) \le l_j + h} depth(v) - \sum_{i=1}^{k} \sum_{l_j + c_i < depth(u) \le l_j + h} depth(u) \\ &= \int_{l_j}^{l_j + h} x \, dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_j + h - c_i} (x + c_i) \, dF(x). \end{aligned}$$

Integrating the last equation by parts gives

$$\begin{aligned} C(T_{a_j}) &= (l_j + h)F(l_j + h) - l_j F(l_j) - \sum_{i=1}^{k} (l_j + h) F(l_j + h - c_i) \\ &\quad + \sum_{i=1}^{k} (l_j + c_i) F(l_j) - \int_{l_j}^{l_j + h} F(x) \, dx + \sum_{i=1}^{k} \int_{l_j}^{l_j + h - c_i} F(x) \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} C(T_{a_j}) &= (l_j + h) \left[F(l_j + h) - \sum_{i=1}^{k} F(l_j + h - c_i) \right] \\ &\quad + (k-1)(l_j + h) F(l_j) \\ &\quad + \int_{l_j}^{l_j + h} F(x) \, dx + \sum_{i=1}^{k} \int_{l_j}^{l_j + h - c_i} F(x) \, dx \\ &= (l_j + h) a_j - \int_{l_j}^{l_j + h} F(x) \, dx + \sum_{i=1}^{k} \int_{l_j}^{l_j + h - c_i} F(x) \, dx. \end{aligned} \tag{28}$$

□

6.2. *The Irrational Case.* We now prove Theorem 5 under the assumption that $(c_1, \ldots, c_r)$ is irrationally related. Recall from Theorem 2 that in this case

$$F(x) = \frac{1}{c \ln \varphi} \varphi^x + o(\varphi^x) \tag{29}$$

and from Lemma 2

$$l_{j+1} - l_j = o(1). \tag{30}$$

Now set $K = (1/(c \ln \varphi))(\varphi^h - \sum_{i=1}^{k} \varphi^{h - c_i} + (k-1))$. Then

LEMMA 12. *If $(c_1, \ldots, c_r)$ is irrationally related and j is such that $a_j \le n < a_{j+1}$, then*

$$a_j = \varphi^{l_j} K + o(\varphi^{l_j}), \tag{31}$$

(32) $$a_{j+1} - a_j = o(\varphi^{l_j}),$$

(33) $$l_j = \log_\varphi n - \log_\varphi K + o(1),$$

(34) $$n = \Theta(a_j) = \Theta(a_{j+1}),$$

(35) $$C(T_{a_j}) = (l_j + h)a_j - \frac{a_j}{\ln \varphi} - o(\varphi^{l_j}).$$

PROOF. From (29)

$$\begin{aligned} a_j &= \int_{l_j}^{l_j+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} dF(x) \\ &= \frac{1}{c \ln \varphi} \varphi^{l_j} \left(\varphi^h - 1 - \sum_{i=1}^{k} (\varphi^{h-c_i} - 1) + o(\varphi^{l_j}) \right) \\ &= \varphi^{l_j} K + o(\varphi^{l_j}). \end{aligned}$$

Combining this with the fact that $l_{j+1} - l_j = o(1)$ yields

$$\begin{aligned} a_{j+1} - a_j &= \varphi^{l_j} K (\varphi^{l_{j+1}-l_j} - 1) + o(\varphi^{l_j}(1 + \varphi^{l_{j+1}-l_j})) \\ &= o(\varphi^{l_j}). \end{aligned}$$

Since $a_j \le n < a_{j+1}$ we have that

(36) $$\varphi^{l_j} K + o(\varphi^{l_j}) \le n < \varphi^{l_{j+1}} K + o(\varphi^{l_{j+1}}).$$

Taking $\log_\varphi$ and again using the fact that $l_{j+1} - l_j = o(1)$ yields

$$l_j = \log_\varphi n - \log_\varphi K + o(1).$$

From (31) and (32) we know that $\Theta(a_j) = \Theta(a_{j+1})$. Since $a_j \le n < a_{j+1}$ this implies that

$$n = \Theta(a_j) = \Theta(a_{j+1}).$$

Finally notice that

$$\begin{aligned} &\int_{l_j}^{l_j+h} F(x)\,dx - \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} F(x)\,dx \\ &\qquad = \frac{1}{\ln \varphi} \varphi^{l_j} \left(\varphi^h - 1 - \sum_{i=1}^{k} (\varphi^{h-c_i} - 1) + o(\varphi^{l_j}) \right) \\ &\qquad = \frac{a_j}{\ln \varphi} + o(\varphi^{l_j}) \end{aligned}$$

so

$$\begin{aligned} C(T_{a_j}) &= (l_j + h)a_j - \int_{l_j}^{l_j+h} F(x)\,dx + \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} F(x)\,dx \\ &= (l_j + h)a_j - \frac{a_j}{\ln \varphi} - o(\varphi^{l_j}). \end{aligned}$$ □

The last lemma we need before proving the result is

LEMMA 13. *Let* $(c_1, \ldots, c_r)$ *be irrationally related and let* j *be such that* $a_j \le n < a_{j+1}$. *Then*

$$C(T_n) = (l_j + h)n - \frac{a_j}{\ln \varphi} + o(n).$$

PROOF. The proof proceeds by splitting into the three cases treated by Theorem 1.

- $a_j \le n \le b_j$. In this case the optimal tree T_n is formed by starting with T_{a_j} and adding the $n - a_j$ new leaves $u_1, u_2, u_3, \ldots, u_{n-a_j}$. By definition, $l_j + h < depth(u_i) \le l_{j+1} + h$ so, from Lemma 10,

$$\begin{aligned} C(T_n) &= C(T_{a_j}) + (n - a_j)(l_j + h) + O(n - a_j)(l_{j+1} - l_j) \\ &= (l_j + h)a_j - \frac{a_j}{\ln \varphi} - o(\varphi^{l_j}) + (n - a_j)(l_j + h) + O(n - a_j)(l_{j+1} - l_j) \\ &= (l_j + h)n - \frac{a_j}{\ln \varphi} + o(n). \end{aligned}$$

  In particular,

$$C(T_{b_j}) = (l_j + h)b_j - \frac{a_j}{\ln \varphi} + o(n).$$

- $n = b_j + p(k-1) < a_{j+1}$. In this case, using the second part of Lemma 10,

$$\begin{aligned} C(T_n) &= C(T_{b_j}) + p(k-1)(l_{j+1} + h) \\ &= (l_j + h)b_j - \frac{a_j}{\ln \varphi} + o(n) + p(k-1)(l_{j+1} + h) \\ &= (l_j + h)n - \frac{a_j}{\ln \varphi} + o(n). \end{aligned}$$

- $n = b_j + p(k-1) + q < a_{j+1}, q \ne 0$. From the third part of Lemma 10,

$$\begin{aligned} C(T_n) &= C(T_{b_j}) + (p(k-1) + q)(l_{j+1} + h) + o(1) \\ &= (l_j + h)b_j - \frac{a_j}{\ln \varphi} + o(n) + (p(k-1) + 1)(l_{j+1} + h) \\ &= (l_j + h)n - \frac{a_j}{\ln \varphi} + o(n). \end{aligned}$$

□

We have just seen that

$$C(T_n) = (l_j + h)n - \frac{a_j}{\ln \varphi} + o(n).$$

To transform this into the form given in the statement of Theorem 5 we note from Lemma 12 that $l_j = \log_\varphi n - \log_\varphi K$. From the same lemma we also have that $a_{j+1} - a_j = o(\varphi^{l_j})$ so if $a_j \le n < a_{j+1}$, then $n - a_j = o(\varphi^{l_j}) = o(n)$. Combining these facts proves that, for all n,

$$\boxed{C(T_n) = n \log_\varphi n + \left(h - \log_\varphi K - \frac{1}{\ln \varphi}\right) n + o(n)}$$

and we have completed the proof for the irrational case.

6.3. *The Rational Case.* We now analyze the rational case. We assume that the c_i's are all positive integers with $\gcd(c_1, \ldots, c_r) = 1$. At the end of this section we quickly discuss what happens when this is not the case.

We note that, by Lemma 2, the assumption of $\gcd(c_1, \ldots, c_r) = 1$ implies that $\exists J$ such that, $\forall j > J$, $l_{j+1} - l_j = 1$. We start the analysis by using this fact to specialize Lemma 10 for the rational case:

LEMMA 14. *For $a_j \le n \le a_{j+1}$,*

$$C(T_n) = C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b_j)\{h\} + O(1),$$

where

$$\mathcal{X}(\theta) = \begin{cases} \theta, & \text{if} \quad \theta > 0; \\ 0, & \text{otherwise}. \end{cases}$$

PROOF. We assume that j is large enough so that $l_{j+1} = l_j + 1$. Otherwise, the $O(1)$ term in the expression will absorb the cost of the tree. There are three cases:

1. $a_j \le n \le b_j$. From (21) of Lemma 10,

$$C(T_n) = C(T_{a_j}) + \sum_{i=1}^{n-a_j} depth(u_i),$$

where the u_i are in the set

$$B_j \backslash A_j = \{v \in LEAF(V_{m_j})\colon\ l_j + h < depth(v) \le l_{j+1} + h\}.$$

Using the fact that $l_{j+1} = l_j + 1$ we have that, $\forall i,\ depth(u_i) = l_j + 1 + \lfloor h \rfloor$. Thus

$$C(T_n) = C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor). \tag{37}$$

In particular,

$$C(T_{b_j}) = C(T_{a_j}) + (b_j - a_j)(l_j + 1 + \lfloor h \rfloor). \tag{38}$$

2. $b_j < n < a_{j+1}$ *with* $n = b_j + p(k-1) + q$, $0 \le q < (k-1)$. From (23) and (24) of Lemma 10,

$$C(T_n) = C(T_{b_j}) + (p(k-1) + q)(l_{j+1} + h) + O(1).$$

Combining this with (38),

$$\begin{aligned} C(T_n) &= C(T_{b_j}) + (p(k-1) + q)(l_{j+1} + h) + O(1) \\ &= C(T_{a_j}) + (b_j - a_j)(l_j + 1 + \lfloor h \rfloor) + (p(k-1) + q)(l_{j+1} + h) + O(1) \\ &= C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + (n - b_j)\lfloor h \rfloor + O(1) \\ &= C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b_j)\{h\} + O(1). \end{aligned}$$

3. $n = a_{j+1}$. Recall from Theorem 1 that $T_{a_{j+1}}$ is obtained from T_{b_j} by making all leaves of depth l_{j+1} into internal nodes with k children. Since $h = (1/(k-1))\sum_{i=1}^{k} c_i$, we have

$$\begin{aligned} C(T_{a_{j+1}}) &= C(T_{b_j}) + \frac{a_{j+1} - b_j}{k-1}\left(\sum_{i=1}^{k} c_i + (k-1)l_{j+1}\right) \\ &= C(T_{a_j}) + (b_j - a_j)(l_j + 1 + \lfloor h \rfloor) + (a_{j+1} - b_j)(l_{j+1} + h) \\ &= C(T_{a_j}) + (a_{j+1} - a_j)(l_j + 1 + \lfloor h \rfloor) + (a_{j+1} - b_j)\{h\} \qquad (39) \\ &= C(T_{a_j}) + (a_{j+1} - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(a_{j+1} - b_j)\{h\}. \qquad (40) \end{aligned}$$ □

To employ the previous lemma successfully we need better expressions for a_j, b_j, and $C(T_{a_j})$.

LEMMA 15. *Let $j \geq 0$ be an integer. Then*

$$\begin{aligned} a_j &= K\varphi^{l_j} + O(\rho^{l_j}), \\ b_j &= K\varphi^{l_j + R} + O(\rho^{l_j}), \\ C(T_{a_j}) &= (l_j + 1 + \lfloor h \rfloor)a_j - \frac{1}{\varphi - 1}K\varphi^{l_j+1} + \{h\}\frac{k-1}{c(1-\varphi^{-1})}\varphi^{l_j} + O(\rho^{l_j}), \end{aligned}$$

where $\rho < \varphi$ and K, R, and A are as defined in Theorem 5:

$$K = \frac{1}{c(1-\varphi^{-1})}\left(\varphi^{\lfloor h \rfloor} - \sum_{i=1}^{k} \varphi^{\lfloor h \rfloor - c_i} + (k-1)\right),$$

$$A = \frac{(k-1)}{c(1-\varphi^{-1})K}, \qquad R = \log_{\varphi}((1-A)\varphi + A).$$

PROOF. Recall from Theorem 2 that

$$F(l_j) = \frac{1}{c(1-\varphi^{-1})}\varphi^{l_j} + O(\rho^{l_j}) \qquad (41)$$

for some $\rho < \varphi$. Plugging this into (25) in Lemma 11 we find that

$$\begin{aligned} a_j &= \int_{l_j}^{l_j+h} dF(x) - \sum_{i=1}^{k}\int_{l_j}^{l_j+h-c_i} dF(x) \\ &= F(l_j + h) - F(l_j) - \sum_{i=1}^{k}(F(l_j + h - c_i) - F(l_j)) \\ &= F(l_j + h) + (k-1)F(l_j) - \sum_{i=1}^{k} F(l_j + h - c_i) \\ &= K\varphi^{l_j} + O(\rho^{l_j}). \end{aligned}$$

Again from Theorem 2

$$\begin{aligned}
b_j &= \int_{l_j}^{l_{j+1}+h} dF(x) - \sum_{i=1}^{k} \int_{l_j}^{l_{j+1}+h-c_i} dF(x) \\
&= F(l_j+1+h) - F(l_j) - \sum_{i=1}^{k} (F(l_j+1+h-c_i) - F(l_j)) \\
&= F(l_j+1+h) + (k-1)F(l_j) - \sum_{i=1}^{k} F(l_j+1+h-c_i) \\
&= K\varphi^{l_j+1} - \frac{1}{c(1-\varphi^{-1})}(\varphi^{j+1} - \varphi^{j}) + O(\rho^{l_j}) \\
&= K\varphi^{l_j}\left(\left(1 - \frac{(k-1)}{c(1-\varphi^{-1})K}\right)\varphi + \frac{c(k-1)}{c(1-\varphi^{-1})K}\right) + O(\rho^{j}) \\
&= K\varphi^{j+R} + O(\rho^{j}).
\end{aligned}$$

Finally, returning yet again to Theorem 2, recall that

$$C(T_{a_j}) = (l_j+h)a_j - \int_{l_j}^{l_j+h} F(x)\,dx + \sum_{i=1}^{k} \int_{l_j}^{l_j+h-c_i} F(x)\,dx. \tag{42}$$

To proceed we need the fact that $F(x)$ only changes at integral values $x = l_j$. Thus, $\forall x,\ F(x) = F(\lfloor x \rfloor)$ and, for any $\delta > 1$, we have

$$\int_{l_j}^{l_j+\delta} F(x)\,dx = \sum_{t=l_j}^{l_j+\lfloor\delta\rfloor-1} F(t) + \{\delta\}F(l_j+\delta).$$

Continuing yields that

$$\begin{aligned}
&-\int_{l_j}^{l_j+h} F(x)\,dx + \sum_{i=1}^{k}\int_{l_j}^{l_j+h-c_i} F(x)\,dx \\
&\quad= -\sum_{t=l_j}^{l_j+\lfloor h\rfloor-1} F(t) + \sum_{i=1}^{k}\sum_{t=l_j}^{l_j+\lfloor h\rfloor-c_i-1} F(t) - \{h\}\left[F(l_j+h) - \sum_{i=1}^{k} F(l_j+h-c_i)\right] \\
&\quad= (1-\{h\})a_j - \sum_{t=l_j+1}^{l_j+\lfloor h\rfloor} F(t) + \sum_{i=1}^{k}\sum_{t=l_j+1}^{l_j+\lfloor h\rfloor-c_i} F(t) + \{h\}(k-1)F(l_j).
\end{aligned}$$

Substituting this back into (42) gives

$$\begin{aligned}
C(T_{a_j}) &= (l_j+1+\lfloor h\rfloor)a_j - \sum_{t=l_j+1}^{l_j+\lfloor h\rfloor} F(t) \\
&\quad + \sum_{i=1}^{k}\sum_{t=l_j+1}^{l_j+\lfloor h\rfloor-c_i} F(t) + \{h\}(k-1)F(l_j).
\end{aligned} \tag{43}$$

Now, using the expansion for $F(x)$ in (41),

$$\begin{aligned}
C(T_{a_j}) &= (l_j + 1 + \lfloor h \rfloor)a_j - \sum_{t=l_j j+1}^{l_j+\lfloor h \rfloor} F(t) + \sum_{i=1}^{k} \sum_{t=l_j j+1}^{l_j+\lfloor h \rfloor - c_i} F(t) + \{h\}(k-1)F(j) \\
&= (l_j + 1 + \lfloor h \rfloor)a_j - \frac{1}{c(1-\varphi^{-1})} \sum_{t=l_j+1}^{l_j+\lfloor h \rfloor} \varphi^t \\
&\quad + \frac{1}{c(1-\varphi^{-1})} \sum_{i=1}^{k} \sum_{t=l_j+1}^{l_j+\lfloor h \rfloor - c_i} \varphi^t + \{h\} \frac{k-1}{c(1-\varphi^{-1})} \varphi^{l_j} + O(\rho^{l_j}) \\
&= (l_j + 1 + \lfloor h \rfloor)a_j - \frac{1}{c(1-\varphi^{-1})} \frac{\varphi^{l_j+\lfloor h \rfloor+1} - \varphi^{l_j+1}}{\varphi - 1} \\
&\quad + \frac{1}{c(1-\varphi^{-1})} \sum_{i=1}^{k} \frac{\varphi^{l_j+\lfloor h \rfloor - c_i+1} - \varphi^{l_j+1}}{\varphi - 1} \\
&\quad + \{h\} \frac{k-1}{c(1-\varphi^{-1})} \varphi^{l_j} + O(\rho^{l_j}) \\
&= (l_j + 1 + \lfloor h \rfloor)a_j - \frac{1}{\varphi - 1} K \varphi^{l_j+1} + \{h\} \frac{k-1}{c(1-\varphi^{-1})} \varphi^{l_j} + O(\rho^{l_j}),
\end{aligned}$$

proving the lemma. □

The last piece we need is a crude bound on the difference in costs between trees. Essentially it says that adding a node at the l_jth level will contribute a cost of $O(l_j)$.

LEMMA 16. *Let j be such that $a_j < n_2 \le a_{j+1}$ and suppose $n_1 < n_2$. Then*

$$C(T_{n_2}) - C(T_{n_1}) = O(l_j(n_2 - n_1)).$$

PROOF. We first assume that $n_1 = n$, $n_2 = n + 1$. This implies that $a_j \le n < n + 1 \le a_{j+1}$ so, by Lemma 14, we have

$$C(T_{n+1}) - C(T_n) = (j + 1 + \lfloor h \rfloor) + \mathcal{X}(n + 1 - b_j) - \mathcal{X}(n - b_j) + O(1).$$

Since $\mathcal{X}(n + 1 - b_j) - \mathcal{X}(n - b_j) \le 1$,

$$C(T_{n+1}) - C(T_n) = O(l_j).$$

Therefore, for $a_j < n_2 \le a_{j+1}$ and $n_1 < n_2$,

$$\begin{aligned}
C(T_{n_2}) - C(T_{n_1}) &= \sum_{n=n_1}^{n_2-1} (C(T_{n+1}) - C(T_n)) \\
&= (n_2 - n_1) O(l_j) \\
&= O(l_j(n_2 - n_1)).
\end{aligned}$$

□

Combining the previous facts permits us to prove the correctness of the theorem for rationally related $(c_1, \ldots, c_r)$ such that the c_i's are all positive integers with $\gcd(c_1, \ldots, c_r) = 1$. To do this recall that, from Lemma 15, we know that $a_j = K\varphi^{l_j} + O(\rho^j)$. Set

$$a'_j = K\varphi^{l_j}. \tag{44}$$

Then $|a_j - a'_j| = O(\rho^j)$. Now suppose $a'_j \le n < a'_{j+1}$. Then

$$l_j = \log_\varphi \frac{a'_j}{K} = \left\lfloor \log_\varphi \frac{n}{K} \right\rfloor, \qquad \varphi^{l_j} = \frac{n}{K}\varphi^{-\{\log_\varphi(n/K)\}}. \tag{45}$$

Again, from Lemma 15, recall that $b_j = K\varphi^{l_j+R} + O(\rho^j)$. Set

$$b'_j = K\varphi^{l_j+R}. \tag{46}$$

Then

$$b'_j = n\varphi^{R-\{\log_\varphi(n/K)\}}, \qquad b_j - b'_j = O(\rho^j). \tag{47}$$

We are assuming that $a'_j \le n < a'_{j+1}$. To prove the theorem we need to treat the three cases

$$a_j \le n \le a_{j+1}, \qquad n < a_j, \qquad a_{j+1} < n$$

separately.

Case 1: $a_j \le n \le a_{j+1}$. From Lemma 14 we have that

$$\begin{aligned} C(T_n) &= C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b_j)\{h\} + O(1) \\ &= C(T_{a_j}) + (n - a'_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b'_j)\{h\} + o(\varphi^{l_j}), \end{aligned}$$

where we are using the fact that

$$n - b_j = n - b'_j + (b_j - b'_j) = n - b'_j + O(\rho^{l_j}).$$

Case 2: $n < a_j$. In this case $a'_j \le n < a_j$, implying that $n - a'_j < a'_j - > a_j = O(\rho^{l_j})$. Thus, using Lemma 16,

$$\begin{aligned} C(T_n) &= C(T_{a_j}) + (C(T_n) - C(T_{a_j})) \\ &= C(T_{a_j}) + O(l_j(n - a'_j)) \\ &= C(T_{a_j}) + O(l_j\rho^{l_j}) \\ &= C(T_{a'_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b'_j)\{h\} + o(\varphi^{l_j}). \end{aligned}$$

Case 3: $a_{j+1} < n$. In this case $a_{j+1} \le n < a'_{j+1}$, implying that $n - a_{j+1} < a'_{j+1} - a_{j+1} = O(\rho^{l_j})$. We attack this in two steps. The first one is quite similar to the previous case:

$$\begin{aligned} C(T_n) &= C(T_{a_{j+1}}) + C(T_n) - C(T_{a_{j+1}}) \\ &= C(T_{a_{j+1}}) + O(l_{j+1}(n - a_{j+1})) \\ &= C(T_{a_{j+1}}) + O(l_j\rho^{l_j}). \end{aligned}$$

Now note that, from Lemma 14,

$$C(T_{a_{j+1}}) = C(T_{a_j}) + (a_{j+1} - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(a_{j+1} - b_j)\{h\} + O(1).$$

Since

$$a_{j+1} - a_j = n - a_j + (a_{j+1} - n) = n - a_j + o(\varphi^{l_j})$$

and

$$a_{j+1} - b_j = n - b'_j + (a_{j+1} - n) + (b_j - b'_j) = n - b'_j + o(\varphi^{l_j})$$

we have

$$C(T_n) = C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b'_j)\{h\} + o(\varphi^{l_j}).$$

Combining cases 1–3 above we find that we have proven, *for all* n, satisfying $a'_j \le n < a'_{j+1}$, that

$$C(T_n) = C(T_{a_j}) + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b'_j)\{h\} + o(\varphi^{l_j}).$$

To complete the theorem we substitute the values for $C(T_{a_j})$, a_j, and b_j found in Lemma 15 and use (16) to find

$$\begin{aligned} C(T_n) &= (l_j + 1 + \lfloor h \rfloor)a_j - \frac{1}{\varphi - 1}K\varphi^{l_j+1} + \{h\}\frac{k-1}{c(1-\varphi^{-1})}\varphi^{l_j} \\ &\quad + (n - a_j)(l_j + 1 + \lfloor h \rfloor) + \mathcal{X}(n - b'_j)\{h\} + o(\varphi^{l_j}) \\ &= (l_j + 1 + \lfloor h \rfloor)n - \frac{1}{\varphi - 1}\varphi^{1-\{\log_\varphi(n/K)\}}n \\ &\quad + \{h\}AK\varphi^{l_j} + \{h\}\mathcal{X}(n - b'_j) + o(\varphi^{l_j}). \end{aligned}$$

To simplify this equation note that $\mathcal{X}(\theta) = \mathcal{X}(-\theta) + \theta$ and that $\mathcal{X}(n\theta) = n\mathcal{X}(\theta)$. Thus

$$\begin{aligned} \{h\}\mathcal{X}(n - b'_j) &= \{h\}\left(n - b'_j + n\mathcal{X}\left(1 - \frac{b'_j}{n}\right)\right) \\ &= \{h\}\left(n - b'_j + n\mathcal{X}\left(1 - \varphi^{R-\{\log_\varphi(n/K)\}}\right)\right). \end{aligned}$$

Also

$$\begin{aligned} AK\varphi^{l_j} - b'_j &= AK\varphi^{l_j} - K\varphi^{l_j+R} \\ &= K\varphi^{l_j}(A - \varphi^R) \\ &= K\varphi^{l_j}[A - ((1-A)\varphi + A)] \\ &= K\varphi^{l_j+1}[1 - A]. \end{aligned}$$

Finally,

$$\begin{aligned} l_j &= \lfloor \log_\varphi(n/K) \rfloor \\ &= \log_\varphi n - \log_\varphi K - \left\{\log_\varphi \frac{n}{K}\right\}. \end{aligned}$$

Combining everything we find that

$$\begin{aligned} C(T_n) &= n\log_\varphi n + \left[h+1-\log_\varphi K - \left\{\log_\varphi \frac{n}{K}\right\} - \frac{1}{\varphi-1}\varphi^{1-\{\log_\varphi(n/K)\}}\right] \\ &\quad + \{h\}K\varphi^{l_j+1}[1-A] + \{h\}\mathcal{X}(1-\varphi^{R-\{\log_\varphi(n/K)\}})n \\ &= n\log_\varphi n + \left[h+1-\log_\varphi K - \left\{\log_\varphi \frac{n}{K}\right\} - \frac{1}{\varphi-1}\varphi^{1-\{\log_\varphi(n/K)\}}\right] \\ &\quad + \{h\}(1-A)\varphi^{1-\{\log_\varphi(n/K)\}} + \{h\}\mathcal{X}(1-\varphi^{R-\{\log_\varphi(n/K)\}})n. \end{aligned}$$

Thus

$$\boxed{C(T_n) = n\log_\varphi n + B\left(\left\{\log_\varphi \frac{n}{K}\right\}\right)n + D\left(\left\{\log_\varphi \frac{n}{K}\right\}\right)n + o(n),}$$

where $B(\theta)$ and $D(\theta)$ are periodic functions with period 1 as defined in the theorem statement.

Recall that we have been assuming that $\gcd(c_1, \ldots, c_r) = 1$. We now quickly discuss what happens if this is not the case. Suppose that $(\overline{c_1}, \ldots, \overline{c_r}) = d(c_1, \ldots, c_r)$ where $\gcd(c_1, \ldots, c_r) = 1$, and $d \neq 1$. Then for every tree T with n leaves for $(c_1, \ldots, c_r)$ there is a corresponding tree $\bar{T}$ for $(\overline{c_1}, \ldots, \overline{c_r})$ with $dC(T) = C(\bar{T})$ and vice versa. The correspondence is the natural one that maps ith edges to ith edges. In particular, if T_n and $\bar{T}_n$ are respective optimal trees, then $C(\bar{T}_n) = d\,C(T_n)$. This is the statement of the theorem so we are done.

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Appendix. Rational versus Irrational Formulas. Theorems 2 and 3 look rather strange and seem to raise more questions than they answer. For example, to what does the difference between rational and nonrational cases combinatorially correspond? Also, why is the expression for $L(x)$ in the rational case so different from that in the irrational one? In the paragraphs that follow we attempt to answer these questions and provide the reader with some intuition as to what is occurring.

First we deal with the combinatorial meaning. Recall that we have previously defined $l_0, l_1, l_2, \ldots$ to be the sequence of values at which $F(x)$ changes. Combinatorially, if the $(c_1, \ldots, c_r)$ are rationally related with $\gcd = d$, then for all j large enough, we find that $l_{j+1} - l_j = d$. For example, if $r = 3$ with $(c_1, c_2, c_3) = (1, \frac{4}{3}, \frac{3}{2}) = \frac{1}{6}(6, 8, 9)$ so $d = \frac{1}{6}$ and $l_0, l_1, l_2, \ldots$ is

$$1,\ \tfrac{4}{3},\ \tfrac{3}{2},\ \tfrac{12}{6},\ \tfrac{14}{6},\ \tfrac{15}{6},\ \tfrac{16}{6},\ \tfrac{17}{6},\ \tfrac{18}{6},\ \tfrac{20}{6},\ \tfrac{21}{6},\ \tfrac{22}{6},\ \tfrac{23}{6},\ \tfrac{24}{6},\ \tfrac{25}{6},\ \ldots.$$

The periodic term $E(x)$ in the equation for $L(x)$ is actually a corrective term that permits us to write an equation valid for all x that still reflects the fact that $L(x)$ only changes at values of the form $x = md$.

If, though, the $(c_1, \ldots, c_r)$ are not rationally related, then the spacing between successive l_j are not regular. For example, if $r = 2$ and $(c_1, c_2) = (1, \sqrt{2})$, then the sequence is

$$1,\ \sqrt{2},\ 2,\ 1+\sqrt{2},\ 2\sqrt{2},\ 3,\ 2+\sqrt{2},\ 1+2\sqrt{2},\ 4,\ 3\sqrt{2},\ 3+\sqrt{2},\ 2+2\sqrt{2},\ \ldots$$

with $\lim(l_{j+1} - l_j) = 0$. Thus for any $\varepsilon > 0$ no matter how small, there exists X such that, $\forall x > X$, $L_\lambda(x - \varepsilon) < L_\lambda(x)$; as x increases, $L(x)$ behaves more and more like a smoothly growing function of x and less and less like a jump function so there is no periodic corrective term.

We can now discuss why the expressions for $L(x)$ (and similarly for $F(x)$) are so different. In order to simplify our statements we throw away the error terms and work with

$$\bar{L}(x) = \begin{cases} E(x)\varphi^x & \text{if } (c_1, \ldots, c_r) \text{ is rationally related,} \\ \dfrac{1-\varphi^{-c_1}}{c \ln \varphi}\varphi^x & \text{otherwise.} \end{cases}$$

We examine the *instantaneous rate of growth* of $\bar{L}(x)$, i.e., how the growth rate of $\bar{L}(x)$ changes with x. If the two types of expressions are really the "same" we expect this value to be the same, irrespective of whether $(c_1, \ldots, c_r)$ is rationally related or not. Note first that if $(c_1, \ldots, c_r)$ is not rationally related, then, since $\bar{L}(x)$ is a differentiable function, the instantaneous rate of growth is simply

$$\bar{L}'(x) = \frac{1-\varphi^{-c_1}}{c \ln \varphi}(\ln \varphi)\varphi^x = \frac{1-\varphi^{-c_1}}{c}\varphi^x.$$

Suppose, though, that $(c_1, \ldots, c_r)$ is rationally related. As discussed above, $L(x)$ will only change at values $x = md$ for integral d, i.e., for $(m-1)d < x < md$ we find $L(x) = L((m-1)d)$ and $L(x)$ has a jump of size $\Theta(\varphi^x)$ at values $x = md$. Thus the derivative of $\bar{L}(x)$ will not exist at these values of x and is zero everywhere else. To capture the instantaneous rate of growth at $x = md$ we must calculate the *average* rate of growth over the interval $((m-1)d, md]$ which will be

$$\frac{1}{d}(\bar{L}(md) - \bar{L}((m-1)d)) = \frac{1}{d}\frac{d(1-\varphi^{-c_1})}{c(1-\varphi^{-d})}(\varphi^{md} - \varphi^{(m-1)d}) = \frac{1-\varphi^{-c_1}}{c}\varphi^x.$$

We therefore see that the two different expressions for $L\bar{(x)}$, one for the rational case and the other for the nonrational one, are the same in the very strong sense that they grow at the same rate. It is actually the requirement that they grow similarly that makes their equations appear different.

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