Algebraic and Combinatorial Properties of the Transfer Matrix of the 2-Dimensional $(1,\infty)$ Runlength Limited Constraint

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Abstract — **We discuss the distributions of eigenvalues, a triangular-matrix decomposition, the inverse and some recurrence relations of the transfer matrix of the two dimensional $(1,\infty)$-*Runlength Limited constrained code.***

I. Introduction

In one dimension a codeword is simply a string over a specified alphabet. A code is a set of codewords. A *(d,k)-Runlength Limited (RLL)* constrained code is the code over the binary alphabet $\{0,1\}$ where the number d (k) is the minimum (maximum) permitted number of 0's separating consecutive 1's in a legal binary sequence $s = s_1 s_2 \cdots$. A 2-dimensional codeword is an array over a specified alphabet and, again, a code is a set of codewords. Recently, there has been a surge of interest in two or higher dimensional constrained codes, e.g., [2] [3] [5] [6] [7]. The 2-D (d,k)-*RLL* constrained code $S^{(2)}_{d,k}$ satisfies the *RLL* constraint both horizontally and vertically. This constraint has also been studied outside of information theory: it is known as the *independent sets of grid graphs problem* in combinatorial graph theory and the *hard square system* or *hard-core lattice gas system* in statistical physics [4] [1]. Besides being interesting in its own right this problem serves as a testbed for developing techniques to study other 2-D constrained ones. Most of the results derived in this paper can be generalized to other such constrained systems.

Let $f(m,n)$ be the number of $m \times n$ arrays that satisfy the constraint. The *capacity* of the constraint is defined to be

$$cap(S) = \lim_{n,m\to\infty} \frac{\log_2 f(m,n)}{nm}.$$

Let C_m $(m \geq 0)$ be the set of all $(m+1)$-vectors v_i of $(0,1)$ that contain no two consecutive $1's$. The number of these is F_{m+3}, the *Fibonacci* number. Let $T_{F_{m+3}} = (t_{ij})$ be an $F_{m+3} \times F_{m+3}$ matrix of $(0,1)$ where $t_{ij} \stackrel{\text{def}}{=} \widehat{v_i v_j}$ to be 1 if the vectors $v_i, v_j \in C_m$ are orthogonal, and 0 otherwise. $T_{F_{m+3}}$ is called the transfer matrix of the constraint, and we have [1]

$$f(m,n) = 1^t \cdot T^n_{F_{m+3}} \cdot 1,$$

where 1 is the vector of which all entries are $1's$. In what follows we will assume that the vectors in C_m are ordered lexicographically.

The standard approach to analyzing the capacity of 2-D constrained systems is to calculate the largest eigenvalues of "big" transfer matrices and then plug these eigenvalues into bounding formulas. In this paper we initiate a study of *properties* of transfer matrices and their eigenvalues.

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II. Main Results

Theorem 1. Let P_m and N_m be the numbers of positive and negative eigenvalues of $T_{F_{m+3}}$. Then

$$P_m - N_m = -\frac{2}{\sqrt{3}} sin\frac{m\pi}{3},$$

where $N_m = N_{m-1} - N_{m-2} + F_{m-2}$ with $N_0 = 1$ and $N_1 = 2$.

$$\text{Let } r_m = \begin{cases} (F_{m+3} + F_{\frac{m+3}{2}})/2, & if\ m\ is\ odd; \\ (F_{m+3} + F_{\frac{m+6}{2}})/2, & if\ m\ is\ even. \end{cases}$$

Theorem 2. Fix m and let $\varphi(\lambda) = det(\lambda I - T_{F_{m+3}})$ be the characteristic polynomial of $T_{F_{m+3}}$. Set $\lambda_1, \lambda_2, \ldots, \lambda_{F_{m+3}}$ to be the eigenvalues of $T_{F_{m+3}}$, sorted in decreasing order. Then $f(m,n)$ is equal to

$$\frac{|\lambda_1 I - T_{11}|}{\varphi'(\lambda_1)}\lambda_1^{n+2} + \frac{|\lambda_{j_2} I - T_{11}|}{\varphi'(\lambda_{j_2})}\lambda_{j_2}^{n+2} + \cdots + \frac{|\lambda_{j_{r_m}} I - T_{11}|}{\varphi'(\lambda_{j_{r_m}})}\lambda_{j_{r_m}}^{n+2},$$

where $T_{F_{m+3}} = \begin{pmatrix} 1 & 1^t \\ 1 & T_{11} \end{pmatrix}$, and all coefficients $\frac{|\lambda_j I - T_{11}|}{\varphi'(\lambda_j)} > 0$. The λ_{j_i} are a subset of size $r_m - 1$ of the F_{m+3} eigenvalues that can be explicitly specified.

Theorem 3. There exist upper triangular matrices $L_{F_{m+3}}$ and diagonal matrices $D_{F_{m+3}}$ (both of which can be specified by recurrence relations) such that $T_{F_{m+3}} = L^t_{F_{m+3}} D_{F_{m+3}} L_{F_{m+3}}$ for all $m > 0$.

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