Unhooking Circulant Graphs: A Combinatorial Method for Counting Spanning Trees and Other Parameters*

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Abstract. It has long been known that the number of spanning trees in circulant graphs with fixed jumps and n nodes satisfies a recurrence relation in n. The proof of this fact was algebraic (relating the products of eigenvalues of the graphs' adjacency matrices) and not combinatorial. In this paper we derive a straightforward combinatorial proof of this fact. Instead of trying to decompose a large circulant graph into smaller ones, our technique is to instead decompose a large circulant graph into different *step graph* cases and then construct a recurrence relation on the step graphs. We then generalize this technique to show that the numbers of Hamiltonian Cycles, Eulerian Cycles and Eulerian Orientations in circulant graphs also satisfy recurrence relations.

1 Introduction

The purpose of this paper is to develop a *combinatorial* derivation of the recurrence relations on the number of spanning trees on circulant graphs. We then extend the technique developed in order to derive recurrence relations on other parameters of circulant graphs.

We start with some definitions and background. The n node *undirected circulant graph* with jumps $s_1, s_2, \ldots s_k$, is denoted by $C_n^{s_1,s_2,\cdots,s_k}$. This is the $2k$ regular graph[1] with n vertices labeled $\{0,\ 1,\ 2,\ \cdots,n-1\}$, such that each vertex i $(0 \le i \le n-1)$ is adjacent to $2k$ vertices $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. The simplest circulant graph is the n vertex cycle C_n^1. The next simplest is the *square of the cycle* $C_n^{1,2}$ in which every vertex is connected to its two neighbors and neighbor's neighbors. Figure 1 illustrates three circulant graphs.

For connected graph G, $T(G)$ denotes the number of spanning trees in G. Counting $T(G)$ is a well studied problem, both for its own sake and because it

* Partially supported by HK CERG grants HKUST6162/00E, HKUST6082/01E and HKUST6206/02E. A full version of this paper is available at [6].

[1] If $\gcd(n, s_1, s_2, \cdots, s_k) > 1$ then the graph is disconnected and contains no spanning trees. Therefore, for the purposes of this extended abstract, we assume that $\gcd(s_1, s_2, \cdots, s_k) = 1$, forcing the graph to be connected. Also note that if $n \le 2s_k$ it is possible that the graph is a *multigraph* with some repeated edges.

J. Hromkovič, M. Nagl, and B. Westfechtel (Eds.): WG 2004, LNCS 3353, pp. 296–307, 2004.

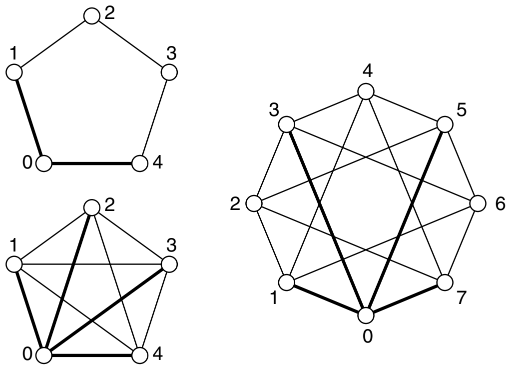

Fig. 1. Three examples of circulant graphs: C_5^1, $C_5^{1,2}$, $C_8^{1,3}$.

has practical implications for network reliability, e.g., [5]. For any *fixed* graph G, Kirchhoff's *Matrix-Tree Theorem* [8] efficiently permits calculating $T(G)$ by evaluating a co-factor of the *Kirchhoff matrix* of G (this essentially calculates the determinant of matrix related to the adjacency matrix of G.)

The interesting problem is in calculating the number of spanning trees in graphs chosen from defined *classes* as a function of a parameter. When G is a circulant graph the behavior of $T(G)$ as a function of n has been well studied. The canonical result is that $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, i.e., $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. This was originally conjectured by Bedrosian [2] and subsequently proven by Kleitman and Golden [9]. The same formula was also conjectured by Boesch and Wang [3] (without the knowledge of [9]). Different proofs can been found in [1, 4, 11]. Formulas for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ are provided in [10]. These were later generalized in [12] to prove the following general theorem: *For any fixed* $1 \leq s_1 < s_2 < \cdots < s_k$,

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2,$$

where a_n *satisfies a recurrence relation of order* 2^{s_k-1} *with constant coefficients.* Knowing the *existence* and *order* of the recurrence relation permits explicitly constructing it by using Kirchhoff's theorem to evaluate $T(C_n^{s_1,s_2,\cdots,s_k})$ for $n = 1, 2, \ldots, 2^{s_k-1}$ and solving for the coefficients of the recurrence relation.

With the exception of that in [9] all of the proofs above work as follows

- Let $s_1, s_2, \ldots s_k$ be fixed.
- Find the eigenvalues of the adjacency matrix of $C_n^{s_1,s_2,\ldots s_k}$. This can be done because the adjacency matrix is a *circulant matrix* and eigenvalues of circulant matrices are well understood.
- Express $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a product function of these eigenvalues.
- Simplify this product to show that $\sqrt{T\left(C_n^{s_1,s_2,\cdots,s_k}\right)/n}$, as a function of n, satisfies a recurrence relation of the given order.

The major difficulty with this technique is that, even though it proves the *existence* of the proper order recurrence relation, it does not provide any combinatorial interpretation, e.g., some type of inclusion-exclusion counting argument, as to why this relation is correct.

As mentioned above, Kleitman and Golden's derivation of $T(C_n^{1,2}) = nF_n^2$, in [9] is an exception to this general technique; their proof is a very clever, fully combinatorial one. Unfortunately, it is also very specific to the special case $C_n^{1,2}$ and can not be extended to cover any other circulant graphs. The major impediment to deriving a general combinatorial proof is that, at first glance, it is difficult to see how to decompose $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ in terms of $T\left(C_m^{s_1,s_2,\cdots,s_k}\right)$ where $m < n$; larger circulant graphs just do not seem to be able to be decomposed into smaller ones.

The main motivation of this paper was to develop a *combinatorial* derivation of the fact that $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$, as a function of n, satisfies a recurrence relation. Our general technique is *unhooking*, i.e., removing all edges

$$\{(i,j)\ :\ n - s_k \le i < n \text{ and } 0 \le j < s_k\}$$

from the graph, creating a new *step graph* $L_n^{s_1,s_2,\cdots,s_k}$. We then define a *fixed* number of classes of forests of $L_n^{s_1,s_2,\cdots,s_k}$ and *combinatorially* derive a system of recurrences counting the number of forests in each class. We then relate this to the original problem by writing $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a linear combination of the number of forests in each class. Technically, we define a $(m \times 1)$-vector (m, the number of forest classes, will be defined later) $\boldsymbol{T}(L_n^{s_1,s_2,\cdots,s_k})$ denoting the number of forests in each class; a $m \times m$ matrix A denoting the system of recurrence relations; and a $(1 \times m)$ row vector $\boldsymbol{\beta}$ such that

$$T\left(C_n^{s_1,s_2,\cdots,s_k}\right) = \boldsymbol{\beta}\cdot\boldsymbol{T}(L_n^{s_1,s_2,\cdots,s_k}), \quad \text{and} \quad \boldsymbol{T}(L_n^{s_1,s_2,\cdots,s_k}) = A\cdot\boldsymbol{T}\big(L_{n-1}^{s_1,s_2,\cdots,s_k}\big).$$

Given these matrix equations, standard techniques, e.g., solving for the generating functions, permit us to derive an order m constant coefficient recurrence relation for $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$.

This technique of unhooking circulant graphs, i.e., developing a system of recurrences on the resultant step graphs and then writing the final result as a function of the step-graph values, is actually quite general and can be used to enumerate many other parameters of circulant graphs. In this extended abstract, we further describe how it can be used to derive recurrence relations for the number of Hamiltonian cycles. In the full version of this paper we also describe how to derive recurrence relations for Eulerian cycles and Eulerian Orientations as well. To the best of our knowledge, this is the first time that techniques for deriving recurrence relations for these other functions of circulant graphs have been developed.

The remainder of the paper is structured as follows. In the first part of section 2 we use our unhooking technique to re-derive the formula $T(C_n^{1,2}) = nF_n^2$. This introduces all of the basic ideas and techniques which are then generalized into a technique for deriving recurrence relations for all $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a function of n. In section 3 we discuss Hamiltonian cycles. Finally, in Section 4, we conclude some comments and open questions.

2 Counting Spanning Trees

2.1 Analyzing $T(C_n^{1,2})$

Let $C_n^{s_1,s_2,\ldots,s_k} = (V, E_C)$ be a circulant graph; $V = \{0, 1, \ldots, n-1\}$ and $E_C = \{(i,j) : i - j \bmod n \in \{s_1, s_2, \ldots, s_k\}\}$.

The associated *Step Graph* $L_n^{s_1,s_2,\ldots,s_k}$ is defined by $L_n^{s_1,s_2,\ldots,s_k} = (V, E_L)$ where $E_L = \{(i,j) : i-j \in \{s_1, s_2, \ldots, s_k\}\}$. For example, the difference between $C_5^{1,2}$ and $L_5^{1,2}$ is $E_C - E_L = \{\{0,4\}, \{0,3\}, \{1,4\}\}$ (See Figure 2).

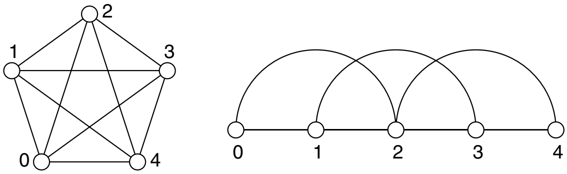

Fig. 2. $C_5^{1,2}$ and $L_5^{1,2}$.

The step graph can be thought of as being obtained from the circulant graph by unhooking the edges that cross over the interval $(n-1, 0)$ in the circulant graph.

For the rest of this subsection we restrict ourselves to the graphs $C_n^{1,2}$ and $L_n^{1,2}$. In the next subsection we will sketch how to generalize the approach to any circulant graph.

The difference between $C_n^{1,2}$ and $L_n^{1,2}$ is the set of edges $E_C - E_L = \{\{0, n-1\}, \{0, n-2\}, \{1, n-1\}\}$ Any spanning tree T of $C_n^{1,2}$ is a collection of $n-1$ edges of E_C; it may or may not contain some edges from $E_C - E_L$.

The main idea behind the counting method is to remove all edges in $E_C - E_L$ from T. Depending upon which edges were in the spanning tree, T can either remain the same or become a disconnected forest of $C_n^{1,2}$. In any case, since we have removed all edges in $E_C - E_L$ what remains is a *forest of* $L_n^{1,2}$ (See Figure 3).

Note that the spanning trees of $C_n^{1,2}$ can be partitioned into eight separate classes, depending upon which, if any of the 3 edges in $E_C - E_L = \{\{0, n-1\}, \{0, n-2\}, \{1, n-1\}\}$ the tree contains. For example, one set of the partition contains all the spanning trees which contain the edge $\{0, n-1\}$ but not $\{0, n-2\}$ and $\{1, n-1\}$. Thus, the number of spanning trees of $C_n^{1,2}$ will be the sum of the numbers of the spanning trees in these eight partitions.

More formally, for $S \subseteq E_C - E_L$ let

$$\mathcal{C}_S(n) = \left\{T \,:\, T \text{ a spanning tree of } C_n^{1,2} \text{ s.t. } T \cap (E_C - E_L) = S\right\}$$

be the set of spanning trees containing only S. Then $T\left(C_n^{1,2}\right) = \sum_S |\mathcal{C}_S(n)|$.

We now examine each set in the partition separately. We take the set previously mentioned again as an example, i.e. $\mathcal{C}_{\{\{0,n-1\}\}}$, in which all trees in the set contain only $\{0, n-1\}$ but not $\{0, n-2\}$ and $\{1, n-1\}$.

After removing $\{0, n-1\}$ each tree in this set becomes a forest in $L_n^{1,2}$ containing exactly two components, one component containing node 0 and the other

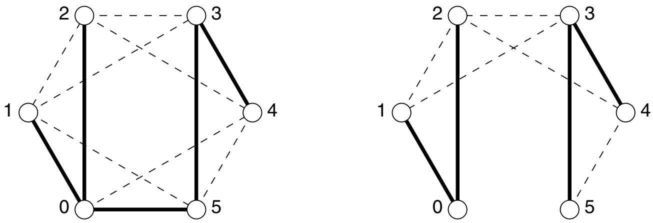

Fig. 3. Removing edges in $E_C - E_L$ from the spanning tree of $C_6^{1,2}$ leaves a disconnected forest of $L_6^{1,2}$. Solid edges are the ones in the tree; dashed ones are existing edges not in the tree. The spanning tree illustrated on the left is in the set $\mathcal{C}_{\{\{0,n-1\}\}}$. The forest on the right is a member of $F_{\{0,1\}\{n-1,n-2\}}(n)$.

containing node $n-1$. These can be further divided into the following four classes of forests with two components in $L_n^{1,2}$:

1. one component contains node 0, the other contains $1, n-2, n-1$
2. one component contains node $0, 1$ the other contains $n-2, n-1$
3. one component contains node $0, n-2$, the other contains $1, n-1$
4. one component contains node $0, 1, n-2$, the other contains $n-1$

This partition is reversible; that is, by adding edge $\{0, n-1\}$ to any of these forests we create the corresponding spanning tree of $C_n^{1,2}$. Thus, summing up the number of forests in the four classes gives us exactly the number of spanning trees of $C_n^{1,2}$ that contain $\{0, n-1\}$ but not $\{0, n-2\}$ and $\{1, n-1\}$.

Extending the above example note that removing all edges in $E_C - E_L = \{\{0, n-1\}, \{0, n-2\}, \{1, n-1\}\}$ from a spanning tree of $C_n^{1,2}$ will result in a forest of $L_n^{1,2}$ that contains 1, 2, 3 or 4 components such that each component (tree) in the forest contains at least one of the four vertices $n-2, n-1, 0, 1$. For later use we will call such forests *legal* and classify the legal forests of $L_n^{1,2}$ by considering how the four vertices are partitioned among the connected components of the forest (we do not consider non-legal forests of $L_n^{1,2}$).

More formally, let $\mathcal{P}$ be the set of partitions of $\{n-2, n-1, 0, 1\}$. For $X \in \mathcal{P}$ define $|X|$ to be the number of sets in X.

Now let $F_X(n)$ be the set containing all forests in $L_n^{1,2}$ with $|X|$ components such that $u, v \in \{n-2, n-1, 0, 1\}$ are in the same component of the forest if and only if they are in the same set of X.

For example, $F_{\{0\}\{1,n-1\}\{n-2\}}(n)$ is the set of spanning forests of $L_n^{1,2}$ with three components s.t. one component contains node 0, another component contains nodes 1 and $n-1$, and the last component contains node $n-2$.

Finally, set $T_X(n) = |F_X(n)|$ to be the number of such forests. Using this notation we can rewrite the discussion above as

$$|\mathcal{C}_{\{0,n-1\}}| = T_{\{0\},\{1,n-2,n-1\}}(n) + T_{\{0,1\},\{n-2,n-1\}}(n) + T_{\{0,n-2\},\{1,n-1\}}(n) + T_{\{0,1,n-2\},\{n-1\}}(n).$$

The important observation here is that if we fix $X \in \mathcal{P}$ and $S \subseteq E_C - E_L$ then adding the set of edges S into a forest in class $F_X(n)$ results in exactly one of the following three consequences and we can determine which of the consequence occurs simply by checking X and S (independent of n)

1. The resulting forest is disconnected.
2. The resulting set of edges contains at least one cycle.
3. The forest becomes a spanning tree of $C_n^{1,2}$ in set $\mathcal{C}_S$.

For example suppose $S = \{\{0, n-1\}, \{0, n-2\}\}$ and

$$X_1 = \{\{0\}, \{1\}, \{n-1\}, \{n-2\}\}, \quad X_2 = \{\{0,1\}, \{n-1, n-2\}\},$$

$$X_3 = \{\{0\}, \{1, n-2\}, \{n-1\}\}.$$

Adding S to a forest in $F_{X_1}(n)$ will leave the forest disconnected; adding S to a forest in $F_{X_2}(n)$ will create a cycle; adding S to a forest in $F_{X_3}(n)$ will create a spanning tree.

We can therefore define

$$\alpha_{S,X} = \begin{cases} 1 \text{ if adding } S \text{ to forest in } F_X(n) \text{ yields a spanning tree} \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

and find that $|\mathcal{C}_S(n)| = \sum_{X \in \mathcal{P}} \alpha_{S,X} T_X(n)$ so

$$T\left(C_n^{1,2}\right) = \sum_S |\mathcal{C}_S(n)| = \sum_{X \in \mathcal{P}} \left(\sum_S \alpha_{S,X} \right) T_X(n). \tag{2}$$

Now define $\boldsymbol{T}\left(L_n^{1,2}\right)$ to be the column vector of all of the $T_X(n)$ ordered as follows:

$$\boldsymbol{T}\left(L_n^{1,2}\right) = \begin{pmatrix} T_{\{0,1,n-2,n-1\}}(n) \\ T_{\{0\}\{1,n-2,n-1\}}(n) \\ T_{\{1\}\{0,n-2,n-1\}}(n) \\ T_{\{n-2\}\{0,1,n-1\}}(n) \\ T_{\{n-1\}\{0,1,n-2\}}(n) \\ T_{\{0,1\}\{n-2,n-1\}}(n) \\ T_{\{0,n-2\}\{1,n-1\}}(n) \\ T_{\{0,n-1\}\{1,n-2\}}(n) \\ T_{\{0\}\{1\}\{n-2,n-1\}}(n) \\ T_{\{0\}\{n-2\}\{1,n-1\}}(n) \\ T_{\{0\}\{n-1\}\{1,n-2\}}(n) \\ T_{\{1\}\{n-1\}\{0,n-2\}}(n) \\ T_{\{1\}\{n-2\}\{0,n-1\}}(n) \\ T_{\{n-2\}\{n-1\}\{0,1\}}(n) \\ T_{\{0\}\{1\}\{n-2\}\{n-1\}}(n) \end{pmatrix}$$

Each entry of $\boldsymbol{T}\left(L_n^{1,2}\right)$ is the number of forests of $L_n^{1,2}$ in the corresponding class. Now, for $X \in \mathcal{P}$ set $\beta_X = \sum_S \alpha_{S,X}$ and $\boldsymbol{\beta} = (\beta_X)_{X \in \mathcal{P}}$. In this notation, (2) simply states that $T(C_n^{1,2}) = \boldsymbol{\beta} \cdot \boldsymbol{T}\left(L_n^{1,2}\right)$. Mechanically working out the values of the β_X from (1) gives

$$T(C_n^{1,2}) = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 3 & 1 & 1 & 2 & 1 \end{pmatrix} \cdot \boldsymbol{T}\left(L_n^{1,2}\right). \tag{3}$$

Until now we have only seen that $T(C_n^{1,2})$ can be written in terms of vector $\boldsymbol{T}\left(L_n^{1,2}\right)$ but this still doesn't say anything about a formula for $T(C_n^{1,2})$. The important observation at this point is that, unlike for circulant graphs, it is quite easy to write a matrix recurrence relation for $\boldsymbol{T}\left(L_n^{1,2}\right)$. In fact, we will be able to write a one-step recurrence of the form $\boldsymbol{T}\left(L_n^{1,2}\right) = A\boldsymbol{T}\left(L_{n-1}^{1,2}\right)$ where A is some fixed integer matrix.

To see this, suppose that we remove node n along with its incident edges from a legal forest in $L_{n+1}^{1,2}$. What remains is a legal forest in $L_n^{1,2}$. We can therefore build all the legal forests of $L_{n+1}^{1,2}$ by knowing the legal forests of $L_n^{1,2}$.

Constructing from the other direction note that the only edges connecting to n in $L_n^{1,2}$ are $\{n, n-1\}$ and $\{n, n-2\}$. Suppose that we add node n and a set of edges $U \subseteq \{\{n, n-1\}, \{n, n-2\}\}$ to a forest of $L_n^{1,2}$ in class $F_X(n)$. The resulting graph will either have a cycle or be a forest in a particular class $F_{X'}(n+1)$ where X' is only determined by X and U (See Figure 4).

Let us now define

$$a_{X',X} = |\{U \subseteq \{\{n, n-1\}, \{n, n-2\}\} : \text{adding } U \text{ to } F_X(n) \text{ yields } F_{X'}(n+1)\}| \quad (4)$$

to be the number of different sets U that can be added to a forest in $F_X(n)$ to yield a forest in $F_{X'}(n+1)$. These $a_{X,X'}$ (which are independent of n) can be mechanically calculated by checking all cases.

Then $T_{X'}(n+1) = \sum_X a_{X',X} T_X(n)$. So, letting $A = (a_{X',X})_{X',X \in \mathcal{P}}$, we find that, for $n \geq 4$, $\boldsymbol{T}\left(L_{n+1}^{1,2}\right) = A\boldsymbol{T}\left(L_n^{1,2}\right)$ and we have derived a system of recurrence relations on the $T_X(n)$.

For our particular case we have worked through the calculations to find A.[2] Combining A with (3) yields a recurrence relation for $T(C_n^{1,2})$. This is a very standard technique so we only sketch the idea here. For all $X \in \mathcal{P}$ create the generating functions $T_X(z) = \sum_n T_X(n) z^n$. $\boldsymbol{T}(L_n^{1,2}) = A\boldsymbol{T}(L_{n-1}^{1,2})$ then corresponds to a system of simultaneous equations *on the generating functions*, and we can use a procedure akin to Gaussian elimination to solve for closed forms of all of the generating functions. Because of the way in which they are derived, all of the generating functions will be rational functions in z, i.e., in the form $P_X(z)/Q_X(z)$ where $P_X(z)$ and $Q_X(z)$ are polynomials in z. Now set $T(z) = \sum_n T(C_n^{1,2}) z^n = \sum_X \beta_X T_X(z)$. As the (weighted) sum of rational functions, $T(z)$ will also be a rational function in z. The fact that $T(z)$ is rational then permits us to recover a recurrence relation on $T(C_n^{1,2})$. Performing the above steps yield

$$T(C_n^{1,2}) = 4T(C_{n-1}^{1,2}) - 10T(C_{n-3}^{1,2}) + 4T(C_{n-5}^{1,2}) - T(C_{n-6}^{1,2})$$

with initial values $36, 125, 384, 1183, 3528, 10404$ for $n = 4, 5, 6, 7, 8, 9$ respectively for which it can be verified that the solution is $T(C_n^{1,2}) = nF_n^2$. We have therefore just given another combinatorial proof of the result due to Kleitman and Golden [9].

[2] The full matrix A is given in [6].

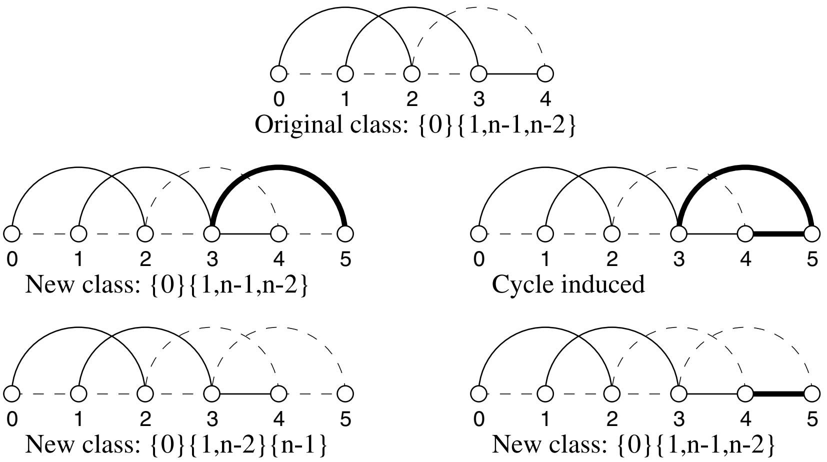

Fig. 4. Different ways to add node 5 to a forest of $L_5^{1,2}$ to generate different classes of forests of $L_6^{1,2}$. Bold edges are the ones added with node 5.

2.2 The General Case

In the previous subsection we developed machinery for counting the number of spanning trees in $C_n^{1,2}$. It is not difficult to see how to generalize this to count the number of spanning trees in $C_n^{s_1,s_2,\ldots,s_k}$. Since this is very similar to the previous section we only sketch the steps.

We start by defining, for all $S \subseteq E_C - E_L$,

$$\mathcal{C}_S(n) = \{T \,:\, T \text{ a spanning tree of } C_n^{s_1,s_2,\ldots,s_k} \text{ s.t. } T \cap (E_C - E_L) = S\}$$

as the set of spanning trees containing only S. Then $T\,(C_n^{s_1,s_2,\ldots,s_k}) = \sum_S |\mathcal{C}_S(n)|$.

Let $W_{s_k} = \{0, 1, \ldots, s_k - 1\} \cup \{n - s_k, n - s_k + 1, \ldots, n - 1\}$. Define $\mathcal{P}_{s_k}$ to be the set of all partitions of W_{s_k}. A *legal forest* of $L_n^{s_1,s_2,\ldots,s_k}$ is one in which every component in the forest contains at least one element in W_{s_k}. For $X \in \mathcal{P}_{s_k}$ define $F_X(n)$ to be the set of all legal forests in $L_n^{s_1,s_2,\ldots,s_k}$ with $|X|$ components such that $u, v \in W_{s_k}$ are in the same component of the forest if and only if they are in the same set of X. Set $T_X(n) = |F_X(n)|$.

We generalize (1) to

$$\alpha_{S,X} = \begin{cases} 1 \text{ if adding } S \text{ to forest in } F_X(n) \text{ yields a spanning tree of } C_n^{s_1,s_2,\ldots,s_k} \\ 0 \text{ otherwise} \end{cases} \tag{5}$$

and find that, as before, $|\mathcal{C}_S(n)| = \sum_{X \in \mathcal{P}_{s_k}} \alpha_{S,X} T_X(n)$ so

$$T\,(C_n^{s_1,s_2,\ldots,s_k}) = \sum_S |\mathcal{C}_S(n)| = \sum_{X \in \mathcal{P}_{s_k}} \left(\sum_S \alpha_{S,X} \right) T_X(n). \tag{6}$$

Let $\boldsymbol{T}\left(L_n^{s_1,s_2,\ldots,s_k}\right)$ be the column vector $\left(T_X(n)\right)_{X\in\mathcal{P}_{s_k}}$, set $\beta_X = \sum_S \alpha_{S,X}$ and define $\boldsymbol{\beta} = (\beta_X)_{X\in\mathcal{P}_{s_k}}$. Then

$$T(C_n^{s_1,s_2,\ldots,s_k}) = \beta \cdot T(L_n^{s_1,s_2,\ldots,s_k}). \tag{7}$$

Exactly as before we can set

$$a_{X',X} = |\{U \subseteq \cup_{i=1}^k \{\{n, n-s_i\}\} \,:\, \text{adding } U \text{ to } F_X(n) \text{ yields } F_{X'}(n+1)\}| \tag{8}$$

and mechanically calculate the $a_{X',X}$ values. Then, letting $A = (a_{X',X})_{X',X\in\mathcal{P}_{s_k}}$, we have for $n \geq 2s_k$,

$$\boldsymbol{T}\left(L_{n+1}^{s_1,s_2,\ldots,s_k}\right) = A\,\boldsymbol{T}\left(L_n^{s_1,s_2,\ldots,s_k}\right). \tag{9}$$

Combining (7) and (9) proves what we want; that $T(C_n^{s_1,s_2,\ldots,s_k})$ can be expressed in terms of a recurrence relation.

3 Hamiltonian Cycles of $C_n^{1,2}$

The unhooking technique developed in the previous section is quite general and can be used to count various other parameters of circulant graphs. In this section we sketch how use it to derive a recurrence relation on the number of Hamiltonian cycles $H(C_n^{1,2})$, in $C_n^{1,2}$. The generalization to deriving a recurrence relation on the number of Hamiltonian cycles $H(C_n^{s_1,s_2,\ldots,s_k})$ in any $C_n^{s_1,s_2,\ldots,s_k}$ will be straightforward.

First note that, as in the spanning tree case, we can partition the Hamiltonian cycles of $C_n^{1,2}$ into eight different classes, depending upon which, if any of the 3 edges in $E_C - E_L = \{\{0, n-1\}, \{0, n-2\}, \{1, n-1\}\}$ the cycle contains.

For $S \subseteq E_C - E_L$ let

$$\mathcal{H}_S(n) = \left\{H \,:\, H \text{ is a Hamiltonian cycle of } C_n^{1,2} \text{ s.t. } H \cap (E_C - E_L) = S\right\}.$$

Then $H\left(C_n^{1,2}\right) = \sum_S |\mathcal{H}_S(n)|$.

Now suppose that we are given some Hamiltonian cycle $H \in \mathcal{H}_S(n)$. After removing the edges in S from H we observe that one of the following three cases must occur:

1. $H - S$ is still a Hamiltonian cycle (of $L_n^{1,2}$).
2. $H - S$ is a Hamiltonian path of $L_n^{1,2}$ with endpoints in $\{0, 1, n-2, n-1\}$
3. $H - S$ is the union of disjoint simple paths in $L_n^{1,2}$ with endpoints in $\{0, 1, n-2, n-1\}$. (See Figure 5).

In the third case, we are considering that if a node is left isolated without any incident edges in $H - S$ then it is in its own path (note that this can only happen to nodes 0 and $n-1$). Also, note that in the second and third case, just by knowing the edges in S it is possible to know what the endpoints of the disjoint paths are (and what, if any, isolated vertices exist).

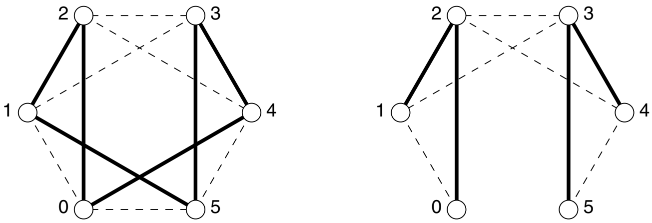

Fig. 5. Decomposition of Hamiltonian cycle $C_6^{1,2}$ to disjoint simple paths in $L_6^{1,2}$.

This observation leads us to define a *legal path decomposition* in $L_n^{1,2}$ to be a disjoint set of paths containing all vertices in V such that all endpoints of the paths are in $\{0, 1, n-2, n-1\}$ and only 0 and $n-1$ are allowed to be isolated vertices. We can classify the legal path decompositions by their endpoints. Define $H_{\{u_1,v_1\},\{u_2,v_2\},\ldots,\{u_w,v_w\}}(n)$ to be the number of subgraphs of $L_n^{1,2}$ with w connected components such that all w components are simple paths with end-points $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_w, v_w\}$ respectively, e.g., $H_{\{1,n-1\}\{0,0\}}(n)$ is the number of all subgraphs of $L_n^{1,2}$ with two components; one component being a path with end-points 1 and $n-1$ and the second component being the single vertex 0. Define one more special case, $H_\emptyset(n)$, to be the number of Hamiltonian cycle of $L_n^{1,2}$. We then define $\boldsymbol{H}(L_n^{1,2})$ to be the column vector:

$$\boldsymbol{H}(L_n^{1,2}) = \begin{pmatrix} H_{\{0,1\}}(n) \\ H_{\{0,n-2\}}(n) \\ H_{\{0,n-1\}}(n) \\ H_{\{1,n-2\}}(n) \\ H_{\{1,n-1\}}(n) \\ H_{\{n-2,n-1\}}(n) \\ H_{\{0,1\}\{n-1,n-1\}}(n) \\ H_{\{0,n-2\}\{n-1,n-1\}}(n) \\ H_{\{1,n-2\}\{0,0\}}(n) \\ H_{\{1,n-2\}\{n-1,n-1\}}(n) \\ H_{\{1,n-2\}\{0,0\}\{n-1,n-1\}}(n) \\ H_{\{1,n-1\}\{0,0\}}(n) \\ H_{\{n-2,n-1\}\{0,0\}}(n) \\ H_{\{0,1\}\{n-2,n-1\}}(n) \\ H_{\{0,n-2\}\{1,n-1\}}(n) \\ H_{\{0,n-1\}\{1,n-2\}}(n) \\ H_\emptyset(n) \end{pmatrix}$$

Let $\mathcal{P}$ be the indices of these items. For $X = \{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_w, v_w\} \in \mathcal{P}$ we say that a legal path decomposition is of type X if it is decomposed into simple paths with end-points $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_w, v_w\}$. For any $S \subseteq E_C - E_L$ and $X \in \mathcal{P}$ define

$$\alpha_{S,X} = \begin{cases} 1 \text{ if adding } S \text{ to path decomposition of type } X \text{ yields a HC} \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

so

$$H\left(C_n^{1,2}\right) = \sum_S |\mathcal{H}_S(n)| = \sum_{X \in \mathcal{P}} \left(\sum_S \alpha_{S,X} \right) H_X(n). \tag{11}$$

Now, for $X \in \mathcal{P}$ set $\beta_X = \sum_S \alpha_{S,X}$ and define $\boldsymbol{\beta} = (\beta_X)_{X\in\mathcal{P}}$. From (11) $H(C_n^{1,2}) = \boldsymbol{\beta} \cdot \boldsymbol{H}\left(L_n^{1,2}\right)$. Evaluating $\boldsymbol{\beta}$ yields

$$H(C_n^{1,2}) = \left(0\;1\;1\;0\;1\;0\;1\;0\;0\;0\;1\;0\;1\;1\;0\;1\;1\right) \cdot \boldsymbol{H}(L_n^{1,2}) \tag{12}$$

Note that adding node n and edge set $U \subseteq \{\{n, n-1\}, \{n, n-2\}\}$ to a legal path decomposition of type X on $L_n^{1,2}$ either does not yield a legal path decomposition or yields a decomposition of type X' on $L_{n+1}^{1,2}$ where X' is fully determined by X and U. Following the ideas in the previous section we therefore define

$$a_{X',X} = |\{U \subseteq \{\{n, n-1\}, \{n, n-2\}\} : \text{adding } U \text{ to decomposition of type } X \text{ yields } X'\}| \tag{13}$$

where $a_{X,X'}$ can be mechanically calculated by checking all cases. Then $H_{X'}(n+1) = \sum_X a_{X',X} H_X(n)$. So, letting $A = (a_{X',X})_{X',X\in\mathcal{P}}$, we find that for $n \geq 4$, $\boldsymbol{H}\left(L_{n+1}^{1,2}\right) = A\,\boldsymbol{H}\left(L_n^{1,2}\right)$. Calculating this A (it appears in [6]), combining with (12) and simplifying as before yields the recurrence

$$H(C_n^{1,2}) = 2H(C_{n-1}^{1,2}) - H(C_{n-3}^{1,2}) - H(C_{n-5}^{1,2}) + H(C_{n-6}^{1,2})$$

with initial values $9, 12, 16, 23, 29, 41$ for $n = 4, 5, 6, 7, 8, 9$ respectively.

Although we only derived a recurrence for $H(C_n^{1,2})$ the technique developed can easily be generalized to derive a recurrence on $H(C_n^{s_1,s_2,\ldots,s_k})$ in much the same way that the technique for calculating $T(C_n^{1,2})$ in Section 2.1 was generalized to calculate $T(C_n^{s_1,s_2,\ldots,s_k})$ in section 2.2. The important changes are (i) to extend the definition of a *legal path decomposition* to $L_n^{s_1,s_2,\ldots,s_k}$ to be a disjoint set of paths containing all vertices in V such that all endpoints of the paths are in $\{0, 1, \ldots, s_k\} \cup \{n - s_k, \ldots, n-2, n-1\}$ and (ii) to set

$$a_{X',X} = |\{U \subseteq \cup_{i=1}^k \{\{n, n-s_i\}\} : \text{adding } U \text{ to decomposition of type } X \text{ yields } X'\}|. \tag{14}$$

Everything else is the same as in the derivation for $H(C_n^{1,2})$ and will yield $H(C_n^{s_1,s_2,\ldots,s_k}) = \boldsymbol{\beta} \cdot \boldsymbol{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right)$ and $\boldsymbol{H}\left(L_{n+1}^{s_1,s_2,\ldots,s_k}\right) = A\,\boldsymbol{H}\left(L_n^{s_1,s_2,\ldots,s_k}\right)$.

4 Conclusion

In this paper we developed the first general *combinatorial* technique for showing that the number of spanning trees in circulant graphs satisfies a recurrence relation. This contrasts to the only previously known general method which used algebraic (spectral) methods.

Our basic approach, unhooking, permits decomposing a problem on *circulant* graphs into many problems on *step* graphs. We then used the fact that step graphs are much more amenable to recursive decomposition to yield our results.

A nice consequence of our technique is that it can be easily modified to work for many other parameters of circulant graphs, e.g., to show that the number of Hamiltonian cycles, Eulerian tours and Eulerian orientations in these graphs also obey a recurrence relation. To the best of our knowledge this is the first time these parameters have been analyzed. We also point out that, even though our technique was described only for *undirected* circulant graphs, it is quite easy to extend it to *directed* circulant graphs as well.

We conclude with an open question. Our analysis implicitly assumed that $s_1, s_2, \ldots, s_k$, the jumps in the circulant graph, are *fixed.* Recent work [7] has shown that in many cases when the s_i are functions of n, then the number of spanning trees also satisfies a recurrence relation. For example, $T(C_{2n}^{1,n}) = \frac{n}{2}[(\sqrt{2}+1)^n + (\sqrt{2}-1)^n]^2$. The proofs of such results are, again, algebraic, involving evaluating products of the eigenvalues of the graph's adjacency matrix. Unfortunately, due to the structure of these graphs, the unhooking technique is not applicable. It is still open as to whether there is any combinatorial derivation of the number of spanning trees of such *non-fixed-jump* circulant graphs.

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