Chapter 29
Limit Theorems for Minimum-Weight Triangulations, Other Euclidean Functionals, and Probabilistic Recurrence Relations (Extended Abstract)

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Abstract

Let MWT(n) be the weight of a minimum-weight triangulation of n points chosen independently from the uniform distribution over $[0,1]^2$. Previous work [11] has shown that $\mathrm{E}(\mathrm{MWT}(n)) = \Theta(\sqrt{n})$.

In this paper we develop techniques for proving that $\frac{\mathrm{MWT}(n)}{\sqrt{n}}$ actually converges to a constant in both expectation and in probability. An immediate consequence is the development of an $O(n^2)$ time algorithm that finds a triangulation whose competive ratio with the MWT is, in a probabilistic sense, exactly one.

The techniques developed to prove the above results are quite general and can also prove the convergence of certain types of probabilistic recurrence equations and other Euclidean Functionals. This is illustrated by using them to prove the convergence of the weight of MWTs of random points in higher dimensions and a sketch of how to use them to prove the convergence of the degree probabilities for Delaunay triangulations in $\Re^2$.

Keywords: Euclidean Linear Functionals, Minimum Weight Triangulations, Limit Theorems

1 Some background

Let S be a finite set of points in $\Re^2$. A *triangulation* of S is a maximum collection of noncrossing edges connecting points in S. This collection of edges partitions the interior of the convex hull of S into a set of triangles $\mathcal{T}$. The *weight* of triangulation $\mathcal{T}$ is the sum of the lengths of the edges contained in $\mathcal{T}$. A *Minimum-Weight Triangulation* of S is one that has has minumum weight among all triangulations of S (Figure 1). We will use MWT(S) to denote both the minimum-weight triangulation and its associated weight; the precise meaning of the notation will be obvious from context.

Finding MWT(S) is a difficult problem for which an efficient solution is still unknown. The best triangulations that can currently be found are at best approximations to the minimal one. For this reason there is interest in knowing exactly what is being approximated, leading in turn to an interest in calculating the average cost of the MWT of random points.

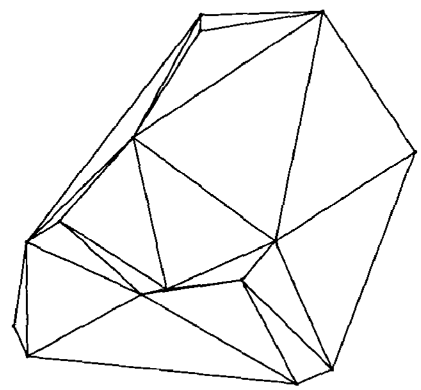

Figure 1: A Minimum Weight Triangulation

It has previously [11] been shown that if $x_1, x_2, \ldots$ are chosen independently and uniformly from the unit square $[0,1]^2$ and $S_n = \{x_1, \ldots, x_n\}$ then $\mathrm{E}(MWT(S_n)) = \Theta(\sqrt{n})$.

This paper was motivated by a desire to prove stronger convergence theorems for MWT(S_n). Doing so required developing new general tools for proving the convergence of linear functionals that might be of independent interest in themselves. More specifically this paper contains the following results:

1. A demonstration of the existence of a constant c such that $\frac{MWT(S_n)}{\sqrt{n}} \to c$ in both expectation and probability. (Section 2)

2. An algorithm that runs in $O(n^2)$ worst-case time on S_n to produce a triangulation PART(S_n) such that $\mathrm{E}\left(\frac{\mathrm{PART}(S_n)}{\sqrt{n}}\right) \to c$ where c is the MWT constant. (Section 2.1)

3. New conditions for proving the convergence of Euclidean Functionals (Section 3.2)

4. Illustration of the use of these new conditions in proving convergence theorems for higher-dimensional "triangulations" of random points in $[0,1]^d$. (Section 3.3) 5points in $[0,1]^2$.

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5. An examination of the convergence of the solutions of certain types of probabilistic recurrence relations that arise quite often in geometric probability. (Section 3.1)

Historically, there is a large literature in the applied-probability community extending back at least to Beardwood, Halton, and Hammersley [1] on how to prove convergence of the type of geometric random variables known as Euclidean Fuctionals. These techniques have been used to examine the asymptotic behavior of a variety of geometric functionals ranging from Travelling Salesmen Tours, to Minimum Spanning Trees, to Matchings and beyond. Surveys of the current state of the art in this type of analysis can be found in [14], [17], and [13]. The existing techniques require that the functionals obey certain growth conditions such as *monotonicity* or *continuity*. For example, the Steiner-Triangulation (ST) problem obeys a monotonicity constraint and $\frac{\mathrm{ST}(S_n)}{\sqrt{n}}$ can therefore be shown to converge to a constant [15] where $ST(S)$ is the weight of the minimum-weight Steiner triangulation of S. Similarly the minimum-spanning tree can be shown to satisfy a continuity type requirement and $\frac{\mathrm{MST}(S_n)}{\sqrt{n}}$ can therefore be shown to converge to a constant [17] where $MST(S)$ is the weight of the minimum-spanning tree of S. The Minimum-Weight triangulation, though, does not satisfy the known conditions and therefore did not fit into the known frameworks leaving the question of its convergence as an open problem.

In this paper we describe a new technique that permits loosening the continuity conditions enough to allow proving convergence for MWTs. We should note that our technique loosens the conditions substantially, allowing proofs of convergence for other problems that were previously unanalyzed. Nothing is free, though and we end up paying for this loosening; our techniques only permit proving convergence in the mean and probability while the older, more restrictive techniques also permitted proving almost-sure convergence.

In the computer science community the literature alluded to above is best known through its use by Karp [7] [8] in the development of a heuristic for finding the Travelling Salesman tour of S_n that, in a probabilistic sense, had an approximation ratio of 1. In a similar fashion we are able to use our proof of the existence of $c > 0$ such that of $\frac{\mathrm{MWT}(S_n)}{\sqrt{n}} \to c$ to develop a polynomial time algorithm for finding a triangulation $\mathrm{PART}(S_n)$ such that $\mathrm{E}\left(\frac{\mathrm{PART}(S_n)}{\sqrt{n}}\right) \to c$ where c is the same constant. (It will even be possible to prove the stronger result that $\frac{\mathrm{PART}(S_n)}{\mathrm{MWT}(S_n)} \to 1$ in expectation and probability but that would be beyond the scope of this extended abstract.) In a probabilistic sense, then, this heuristic has an approximation ratio of 1 improving upon the previously best known heuristics [10] [3] which had only constant approximation ratios greater than 1.

The paper is structured as follows: Section 2 proves some structural properties of Minimum Weight Triangulations (MWTs) and then states (without proof) a convergence theorem for MWTs. It then describes how the convergence theorem implies a triangulation heuristic that, in a probabilistic sense, has an approximation ratio of of one with the MWT. Section 3 proves convergence theorems for certain general types of probabilistic recurrence relations and then describes applications, including the proof of the MWT convergence theorem of Section 2.

To conclude we point out that the techniques developed in this paper, at their core, reduce to the analysis of the solutions to probablistic recurrence relations of a certain type that occur quite naturally in the analysis of many geometric problems. Analyzing these recurrence relations goes part of the way towards resolving an open problem posed by Karp in [9].

Note: In this extended abstract we omit the technical details of the proofs of many of the theorems and lemmas, contenting ourselves with only providing intuition as to why they are correct.

2 The Minimum-Weight Triangulation

Let $x_1, x_2, x_3, \ldots$ be points chosen independently from the uniform distribution over the unit square $[0,1]^2$ and set $S_n = \{x_1, x_2, \ldots, x_n\}$. In this section we derive some basic facts about $\mathrm{MWT}(S_n)$ that, taken together with a general theorem about Euclidean Functionals stated in the next section will imply

THEOREM 1. *There exists a constant $c > 0$ such that $\frac{\mathrm{MWT}(S_n)}{\sqrt{n}}$ converges to c in both expectation and probability, i.e.,*

$$\mathrm{E}\left(\frac{\mathrm{MWT}(S_n)}{\sqrt{n}}\right) \to c,$$

and

$$\forall \epsilon > 0, \quad \Pr\left(\left|\frac{\mathrm{MWT}(S_n)}{\sqrt{n}} - c\right| > \epsilon\right) \to 0.$$

Furthermore

$$\mathrm{VAR}\left(\frac{\mathrm{MWT}(S_n)}{\sqrt{n}}\right) \to 0.$$

The actual proof of Theorem 1 is deferred until the end of section 3.2. We devote the rest of this section to studying the MWT.

LEMMA 2.1. *Let $S \subset [0,1]^2$ be any finite subset with $|S| = n$ and let S_n be a random subset as described above. Then*

1. $\mathrm{MWT}(\emptyset) = 0$ *and* $\mathrm{MWT}(S_n) < n^2$.
2. *For every* $\alpha > 0$, $\mathrm{MWT}(\alpha S) = \alpha \mathrm{MWT}(S)$ *where* $\alpha S = \{\alpha x : x \in S\}$.
3. *For every* $x \in \Re^d$ *we have* $\mathrm{MWT}(S+x) = \mathrm{MWT}(S)$ *where* $S + x = \{y + x : y \in S\}$.
4. *For* $m = 2$ *and* $m = 3$ *consider the partition* $(Q_i)_{i \le m^2}$ *of* $[0,1]^2$ *into* m^2 *equal sized squares. Then*
$$\mathrm{MWT}(S) \le \sum_{i \le m^2} \mathrm{MWT}(S \cap Q_i) + F(S).$$
where
$$\Pr\left(F(S_n) > \ln^2 n\right) = n^{-\Omega(\ln n)}.$$
5. $\mathrm{E}(\mathrm{MWT}(S_{n+1}))$
$$\le \mathrm{E}(MWT(S_n)) + O\left(\frac{\ln^2 n}{\sqrt{n}}\right).$$

Proof The proofs of items 1, 2, and 3 are straightforward. To prove items 4 and 5 we will need to use the fact that the "outside" of *any* triangulation of a set S consists of exactly the edges on $CH(S)$, the convex hull of S (Figure 1). For this reason we will need some well known facts concerning the convex hull of random points:

LEMMA 2.2. *The number of points on the convex hull of* S_n*, satisfies* $\mathrm{E}(|CH(S_n)|) = \Theta(\ln n)$ *and for any fixed* $k > 0$, $\Pr\left(|CH(S_n)| > k \ln^2 n\right) = n^{-\Omega(\ln n)}$.

Proof The expectation result comes from [12]. The high probability bound does not seem to be written down but can be proven using standard Chernoff-bound texhniques. □

We now prove item 4 restricted to the case that $m = 2$; the proof for the case $m = 3$ is almost exactly the same. Note that one way of triangulating S is to first construct $\mathrm{MWT}(S \cap Q_i)$, for $i = 1,2,3,4$, and then add edges connecting points on the convex hulls of the four sets until a full triangulation is found (Figure 2). The number of such edges added is bounded by $2\sum_{i=1}^{4} |CH(S \cap Q_i)|$ while the length of each such edge is at most $\sqrt{2}$. Since the weight of this triangulation upperbounds the weight of $\mathrm{MWT}(S)$ we have shown that

$$\mathrm{MWT}(S) \le \sum_{i \le m^2} \mathrm{MWT}(S \cap Q_i) + F(S).$$

where $F(S) \le 2\sqrt{2}\sum_{i=1}^{4} |CH(S \cap Q_i)|$. The proof of item 4 follows by using Lemma 2.2 to show that $\Pr\left(|CH(S \cap Q_i)| > \ln^2 n/(8\sqrt{2})\right) = n^{-\Omega(\ln n)}$.

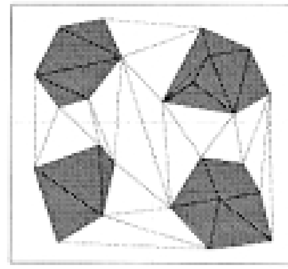

Figure 2: Illustration for the proof of item 4.

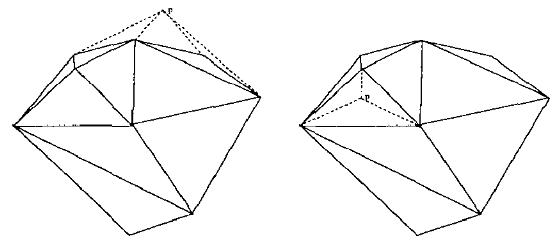

Figure 3: Illustrations for the proof of item 5. The point p is added to a previously existing triangulation $\mathcal{T}$.

To prove item 5 assume for the moment that S_n, along with an arbitrary triangulation $\mathcal{T}$ of S_n, are known (as with the MWT we will let context dictate whether $\mathcal{T}$ is the actual triangulation or its weight). Let $T_1, T_2, \ldots, T_m$ be the triangles comprising $\mathcal{T}$ and set $C_1, C_2, \ldots, C_m$ to be the length of their corresponding perimeters. Finally let x_{n+1} be some point chosen randomly from $[0,1]^2$ and construct a triangulation $\mathcal{T}'$ of $S_{n+1} = S_n \cup \{x_{n+1}\}$ as follows:

- If x_{n+1} is outside $CH(S_n)$ add all edges not intersecting $CH(S_n)$ that connect x_{n+1} to points on $CH(S_n)$. The total length of all edges added is at most $\sqrt{2}|CH(S_n)|$ (left diagram in Figure 2).
- If x_{n+1} is inside $CH(S_n)$ let T_i be the unique triangle containing x_{n+1} and draw the three edges connecting x_{n+1} to the vertices of T_i.[2] The total length of all edges added is at most $3C_i$ (right diagram in Figure 2).

Now define the indicator random variables

$$I_0 = \begin{cases} 1 & x_{n+1} \text{ outside } CH(S_n) \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_i = \begin{cases} 1 & x_{n+1} \text{ inside triangle } T_i \\ 0 & \text{otherwise} \end{cases}$$

[2] We do not consider the zero-probability event of x_{n+1} falling upon some *edge* of $\mathcal{T}$.

Then

$$\mathcal{T}' - \mathcal{T} \leq \sqrt{2} I_0 \cdot |CH(S_n)| + 3 \sum_i I_i C_i.$$

Notice that $\mathrm{E}(I_i) = A_i$ where $A_i = Area(T_i)$. so

$$\text{(2.1)} \; \mathrm{E}(\mathcal{T}') - \mathcal{T} \leq \sqrt{2} \mathrm{E}(I_0 \cdot |CH(S_n)|) + 3 \sum_i A_i C_i.$$

Recall, now that S_n and S_{n+1} are *random* point sets so Lemma 2.2 implies that, with probability $1 - n^{-\Omega(\ln n)}$, we may assume that both $|CH(S_n)| < \ln^2 n$ and $|CH(S_{n+1})| < \ln^2 n$. We now use a standard trick from the analysis of randomized algorithms and notice that $I_0 = 1$ if and only if x_{n+1} is on $CH(S_{n+1})$ which, because x_{n+1} has the same distribution as a random point from S_{n+1}, occurs only with probability $\frac{|CH(S_{n+1})|}{n}$. Thus, from Lemma 2.2,

$$\mathrm{E}(I_0 \cdot |CH(S_n)|) = O(\ln^4 n/n).$$

To analyze the second summand on the right hand side of (2.1) we need the following lemma:

LEMMA 2.3. *Let S_n be as defined above and let $\mathcal{T}$ be any triangulation of S_n. then*

$$\Pr\left(\exists \text{ triangle } T \in \mathcal{T},\ Area(T) > \frac{\ln^2 n}{n}\right) = n^{-\Omega(\ln n)}.$$

Proof For $i < j < k \leq n$ let $T_{i,j,k}$ be the triangle with vertices x_i, x_j, x_k and $A_{i,j,k} = Area(T_{i,j,k})$. If $T_{i,j,k}$ is in *any* triangulation it may not contain any points of S_n in its interior. The probability of $T_{i,j,k}$ having an empty interior is $(1 - A_{i,j,k})^{n-3}$. But, if $A_{i,j,k} > \frac{\ln^2 n}{n}$, then $(1 - A_{i,j,k})^{n-3} = n^{-\Omega(\ln n)}$. Summing over all i, j, k completes the proof. □

Now note that every edge in the triangulation can appear in at most two triangles so $\sum_i C_i \leq 2\mathcal{T}$. Plugging this fact, along with the previous lemma, back into (2.1) we find that, for any triangulation $\mathcal{T}$,

$$\begin{aligned} \mathrm{E}(\mathcal{T}') - \mathcal{T} &\leq O\left(\frac{\ln^4 n}{n}\right) + \frac{3\ln^2 n}{n} \sum_i C_i \\ &\leq \left(\frac{\ln^4 n}{n}\right) + \frac{6 \ln^2 n}{n} \mathcal{T} \end{aligned}$$

Now let $\mathcal{T} = \mathrm{MWT}(S_n)$. From [11] we know that $\mathrm{E}(\mathrm{MWT}(S_n)) = \Theta(\sqrt{n})$. Plugging this into the above equation and taking expectations yields

$$\begin{aligned} &\mathrm{E}(\mathrm{MWT}(S_{n+1})) - \mathrm{E}(\mathrm{MWT}(S_n)) \\ &\leq \mathrm{E}((\mathrm{MWT}(S_n))') - \mathrm{E}(\mathrm{MWT}(S_n)) \\ &\leq O\left(\frac{\ln^4 n}{n}\right) + \frac{6\ln^2 n}{n} O(\sqrt{n}) = O\left(\frac{\ln^2 n}{\sqrt{n}}\right) \end{aligned}$$

completing the proof of Lemma 2.1 □

2.1 A Heuristic Triangulation Algorithm

Let $S \subset [0,1]^2$, $S = n$, and S_n as described above. In this section we describe an $O(n^2)$ worst case time algorithm for finding a triangulation, PART(S), that, in a probablistic sense, closely approximates MWT(S). In fact, using the results of the previous subsection we will sketch a proof that, if S_n is a set of n points chosen randomly from the unit square, then $\mathrm{E}\left(\frac{\mathrm{PART}(S_n)}{\sqrt{n}}\right) \to c$ where c is exactly the constant in Theorem 1 (In reality one can go further and show that $\frac{\mathrm{PART}(S_n)}{\mathrm{MWT}(S_n)} \to 1$ in both expectation and probability but the proof of that statement is beyond the scope of this paper).

To proceed we will need two basic facts. The first, which follows from Theorem 1, is that for every $\epsilon > 0$ there exists N such that $\forall n > N$, $\mathrm{E}(\mathrm{MWT}(S_n)) \leq (c+\epsilon)\sqrt{n}$. The second is the existence of a basic dynamic programming algorithm for finding the MWT of n points in $O(n^{n+2})$ worst case time (see e.g., [4]). Thus, the MWT of $\frac{\ln n}{\ln \ln n}$ points can be found in $O(n)$ worst case time.

Before defining PART(S) we introduce the idea of a k-partition and a k-triangulation.

Given integer $k \geq 0$ the *k-partition* of $[0,1]^2$ is the partition of the unit-square into 4^k smaller squares Q_i^k, $i \leq 4^k$, each of which has area $1/4^k$. We denote by $Q_{i,j}^k$ $j = 1,2,3,4$ the four squares in the $(k+1)$-partition that combine to form Q_i^k.

Now, given $S \subset [0,1]^2$ and $\mathcal{T} \subset S \times S$, let $\mathcal{T}_{Q_i^k}$ be the set of edges in $\mathcal{T}$ that fall totally within Q_i^k. We say that $\mathcal{T}$ is a *k-triangulation* of S if (a) for all $i \leq 4^k$, $\mathcal{T}_{Q_i^k}$ is a triangulation of $Q_i^k \cap S$, and (b) $\mathcal{T} = \bigcup_i \mathcal{T}_{Q_i^k}$. A k-triangulation thus can be thought of as the union of triangulations of the points in each of the 4^k small squares.

Suppose now that we are given some k-triangulation $\mathcal{T}$, of S. This can be extended to a $(k-1)$-triangulation of S as follows: for each $i \leq 4^{k-1}$ construct a triangulation of Q_i^{k-1} by starting with the edges in $\cup_{j=1}^4 \mathcal{T}_{Q_{i,j}^{k-1}}$ and adding arbitrary non-intersecting edges connecting vertices in $\cup_{j=1}^4 CH\left(S \cap Q_{i,j}^{k-1}\right)$ until a triangulation is formed. The total number of such edges added will be at most $2\sum_{j=1}^4 \left|CH\left(S \cap Q_{i,j}^{k-1}\right)\right|$. Using standard techniques one can show that the total number of operations required to extend a k triangulation to a $(k-1)$-triangulation is

$$O(\log n) \sum_{i \leq 4^k} \left|CH\left(S \cap Q_i^k\right)\right| = O(n \log n).$$

The total length of all edges added in this extension will be at most

$$(2.2)\qquad C_k = 2^{-(k-1)}\sqrt{2}\sum_{i\le 4^k}\left|CH\left(S\cap Q_i^k\right)\right|.$$

If $S = S_n$ is a random point set as described above then $|S\cap Q_i^k|$ is binomially distributed with parameters n, $p = 4^{-k}$ so if $np > \sqrt{\ln n}$ Lemma 2.2 tells us that $\mathrm{E}\left(\left|CH\left(S\cap Q_i^k\right)\right|\right) = \Theta\left(\ln\left(4^{-k}n\right)\right)$ and thus $\mathrm{E}\left(C_k\right) = O\left(2^k\ln\left(4^{-k}n\right)\right)$

We can now describe our algorithm. Let $l = \lfloor\log_4 n - \frac{1}{2}\log_4\log_4 n\rfloor$. Construct an l-triangulation $\mathcal{T}^l$ of S by triangulating each set $S\cap Q_i^l$ separately as follows: if $|S\cap Q_i^l| \le \frac{\ln n}{\ln\ln n}$ use a dynamic programming (or some brute force) algorithm to find MWT($S\cap Q_i^l$) in $O(n)$ time. Otherwise find the Delaunay triangulation of $S\cap Q_i^l$ in $O(|S\cap Q_i^l|)\log n$ time. The total time needed to construct this l-triangulation is $O(n^2)$.

Next iteratively construct a sequence $\mathcal{T}^k$, $k = l-1, l-2, \ldots, 0$ of k-triangulations where $\mathcal{T}^k$ is the k-triangulation created by extending the $(k+1)$-triangulation $\mathcal{T}^{k+1}$ using the procedure described previously. Set PART(S) $= \mathcal{T}^0$ By definition PART(S) is a triangulation of S.

Since each extension step requires $O(n\log n)$ time the construction of all of the extensions uses $O(n\log^2 n)$ time and thus the complete construction of PART(S) requires $O(n^2)$ time.

We claim that $\forall\epsilon > 0$, if n is large enough, then $\mathrm{E}\left(\mathrm{PART}(S_n)\right) \le (c+\epsilon)\sqrt{n}$. Because the definition of c requires that $\forall\epsilon > 0$, $\mathrm{E}\left(\mathrm{PART}(S_n)\right) \ge (c-\epsilon)\sqrt{n}$ for all large enough n this will imply $\mathrm{E}\left(\frac{\mathrm{PART}(S_n)}{\sqrt{n}}\right) \to c$.

First note that the expected cost of all edges added in the extension stage of the algorithm is

$$\sum_{k=0}^{l}\mathrm{E}\left(C_k\right) = O\left(\sum_{k=0}^{l}2^k\ln\left(4^{-k}n\right)\right) = o(\sqrt{n})$$

and thus we only need to bound the expected weight of the first stage of the algorithm.

Let $X_i = S_n\cap Q_k^l$. This is a binomial random variable with parameters n and $p = 4^{-l}$ and thus $\mathrm{E}\left(X_i\right) \sim \sqrt{\log_4 n}$. Chernoff-bound techniques prove that $\ln\ln n < X_i < \frac{\ln n}{\ln\ln n}$ with probability $1 - O(n^{-3})$. We may therefore assume that X_i is within this range (because its expected contribution to the weight of the l-triangulation when it is out of the range is $o(\sqrt{np})$). Thus, in our analysis we may assume that the algorithm will construct MWT($S_n\cap Q_i^l$). If n is large enough (so that $\ln\ln n$ is large enough) we have that

$$\begin{aligned}\mathrm{E}\left(\mathrm{MWT}(S_n\cap Q_i^l)\right) &< (c+\frac{\epsilon}{2})\mathrm{E}\left(\sqrt{X_i}\right)\\ &= (c+\frac{\epsilon}{2})\sqrt{np}(1+o(1))\\ &< (c+\epsilon)\sqrt{np}\end{aligned}$$

Thus, for large enough n, the total expected cost of the l-triangulation built will be at most

$$2^l(c+\epsilon)\sqrt{np} = (c+\epsilon)\sqrt{n}$$

completing the analysis.

3 Convergence Theorems

In this section we prove a theorem implying the convergence of many Euclidean functionals including the *MWT*. Before doing so, though, we will have to prove convergence theorems for a form of probabilistic recurrence relation that arises quite often (in a hidden form) in the analyses of the functionals.

3.1 Basic Lemmas

DEFINITION 3.1. *A sequence of positive reals* ϕ_n, $n = 1,2,3,\ldots$ *is* binomially bounded with parameter a *if there exists some* $\alpha\in\Re^+$ *such that* $\phi_n = O(n^\alpha)$ *and*

$$(3.3)\ \phi_n \le \mathrm{E}(\phi_X)\left[1 + O\left(\frac{1}{\ln^2 n}\right)\right] + O\left(\frac{1}{\ln^2 n}\right)$$

where X *is a binomial random variable with parameters* n, $p = 1/a$.

We emphasize here that the actual items in the sequence, the ϕ_n, are *deterministic*; the recurrence relation that they obey though, is *probabilistic*.

The first thing to notice about these sequences is that they are bounded (the proof of the lemma is omitted in this abstract):

LEMMA 3.1. *Let* ϕ_n, $n = 1,2,3,\ldots$ *be a binomially bounded sequence with parameter* $a > 1$. *Then the sequence is bounded, i.e., there exists* c *such that* $\forall n$, $\phi_n \le c$. *Furthermore, for all* n,

$$(3.4)\qquad \phi_n \le v_n + O\left(\frac{1}{\ln^2 n}\right)$$

where $v_n = \max_{|i-\frac{n}{a}|<\sqrt{n}\ln n}\phi_i$.

If the sequence obeys a slightly more stringent condition it can be shown that it is not only bounded but that it also converges. It is this fact which forms the basis for all of the other results in this paper.

THEOREM 2. *Let* ϕ_n, $n = 1,2,3,\ldots$ *be a binomially bounded sequence for* both *parameter* $a > 1$ and *parameter* $b > 1$ *where* $\log_a b$ *is irrational. Furthermore suppose that* ϕ_n *satisfies at least one of the following two conditions:*

$$(A)\ \forall n,\ \phi_{n+1} \le \phi_n + O\left(\frac{1}{\sqrt{n}\ln^2 n}\right),$$

$$(B)\ \forall n\ \phi_n \le \phi_{n+1} + O\left(\frac{1}{\sqrt{n}\ln^2 n}\right).$$

Then there exists $c \ge 0$ *such that* $\lim_{n\to\infty} \phi_n = c$.

Proof

The proof has two parts. Let $\epsilon > 0$. In the first part of the proof we use conditions (A) or (B) to prove the existence of an interval I such that $\forall n \in I$ $|\phi_n - c| \le \epsilon/2$. We then use the definition of binomially bounded sequences to bootstrap this fact to show the existence of N such that $|\phi_n - c| < \epsilon$ for all $n > N$.

The actual proof will follow from two combinatorial lemmas whose proofs will be omitted in this extended abstract. These are:

LEMMA 3.2. *Let* $I = [x, y] \subseteq \Re$, $0 < x < y$, *be some closed interval* a, b *such that* $1 < a < b$ *with* $\log_a b$ *irrational. Then there exists* N' *such that*

$$[N', \infty) \subseteq \bigcup_{j=0}^{\infty}\bigcup_{i=0}^{\infty} (a^i b^j)\, I$$

where $\alpha I = [\alpha x, \alpha y]$.

LEMMA 3.3. *Let* ϕ_n, $n = 1, 2, 3, \ldots$ *be a binomially bounded sequence with both parameter* $a > 1$ *and parameter* $b > 1$. *Set* $\alpha = \frac{6}{1-1/\sqrt{a}} + \frac{18}{1-1/\sqrt{b}}$. *Then there exists* $N' > 0$ *such that the following statement is true for all* $N > N'$: *Let*

$$d_N = \max\{\phi_n \,:\, N \le n \le N + \alpha\sqrt{N}\ln N\}.$$

and

$$I_N = [N + \frac{\alpha}{3}\sqrt{N}\ln N, N + \frac{2\alpha}{3}\sqrt{N}\ln N].$$

Then

$$\forall i, j \ge 0, \quad \forall n \in (a^i b^j)\, I_N, \quad \phi_n \le d_N + O\left(\frac{1}{\ln N}\right). \tag{3.5}$$

We now proceed with the proof of the theorem: Lemma 3.1 tells us that

$$c = \lim_{m\to\infty} \inf_{n\ge m} \phi_n < \infty.$$

For every $N_1 > 0$ and $\epsilon > 0$ we can therefore find $N_2 > N_1$ such that $\phi_{N_2} < c + \epsilon/2$. If ϕ_n satisfies condition (A) set $N = N_2$; if it satisfies condition (B) set $N = N_2 - \alpha\sqrt{N_2}\ln N_2$. In both cases the respective conditions guarantee that if N_1 is chosen large enough then

$$\begin{aligned} d_N &= \max\{\phi_n \,:\, N \le n \le N + \alpha\sqrt{N}\ln N\} \\ &< c + 2\epsilon/3. \end{aligned}$$

Pick N_1 large enough so that for $N \ge N_1$ in equation (3.5) the $O\left(\frac{1}{\ln N}\right)$ term is less than $\epsilon/3$. Then Lemma 3.3 implies that for the chosen $N > N_1$,

$$\forall i, j \ge 0, \quad \forall n \in (a^i b^j)\, I_N, \quad \phi_n < c + \epsilon.$$

Lemma 3.2 then implies the existence of an integer N' such that, for all $n \ge N$, $\phi_n < c + \epsilon$. Thus

$$\lim_{m\to\infty} \sup_{n>m} \phi_n < c + \epsilon = \lim_{m\to\infty} \inf_{n>m} \phi_n + \epsilon.$$

Since this is true for every $\epsilon > 0$ it implies that

$$\lim_{m\to\infty} \sup_{n>m} \phi_n = \lim_{m\to\infty} \inf_{n>m} \phi_n = c = \lim_{n\to\infty} \phi_n$$

completing the proof of the theorem. □

3.2 Convergence of Euclidean Functionals

Let L be a functional mapping finite subsets of $\Re^d$ to positive reals. Also, let $x_1, x_2, x_3, \ldots$ be a sequence of points chosen independently from the uniform distribution over $[0,1]^d$ and set $S_n = \{x_1, \ldots, x_n\}$. We define certain conditions on L.

(A0) $L(\emptyset) = 0$. Furthermore, there exist constants α_1, α_2, such that for every finite subset $S \subset [0,1]^d$, $|S| = n$, we have $L(S) \le \alpha_1 n^{\alpha_2}$.

(A1) There is some constant d' such that for every $\alpha > 0$, and every finite subset $S \subset \Re^d$, $L(\alpha S) = \alpha^{d'} L(S)$ where $\alpha S = \{\alpha x \,:\, x \in S\}$. The constant d' is the *scaling factor* of L.

(A2) For every $x \in \Re^d$ and every finite subset $S \subset \Re^d$ we have $L(S + x) = L(S)$ where $S + x = \{y + x \,:\, y \in S\}$.

(A3) For some integer m consider the partition $(Q_i)_{i\le m^d}$ of $[0,1]^d$ into m^d equal sized cubes. Let $S \subset [0,1]^d$ be any finite subset. Then there exists a functional $F(S)$ such that

$$L(S) \le \sum_{i\le m^d} L(S \cap Q_i) + F(S) \tag{3.6}$$

where

$$\Pr\left(F(S) > \frac{n^{(d-d')/d}}{\ln^2 n}\right) = n^{-\Omega(\ln n)}.$$

(A4) L satisfies one of the two following conditions

$$\forall n\ \mathrm{E}(L(S_n)) \le \mathrm{E}(L(S_{n+1})) + O\left(\frac{n^{(d-d')/d}}{\sqrt{n}\ln^2 n}\right) \tag{3.7}$$

$$\forall n \; \mathrm{E}(L(S_{n+1})) \le \mathrm{E}(L(S_n)) + O\left(\frac{n^{(d-d')/d}}{\sqrt{n}\ln^2 n}\right) \tag{3.8}$$

We call any L that satsifies conditions (A0), (A1), and (A2) a *Euclidean Functional.* Condition (A3) is a *subadditivity*-like condition and (A4) a *continuity*-one. (Note that if L is monotonically increasing then L trivially satisfies the first condition of (A4).)For an understanding of how these conditions compare to the ones existing in the literature in both their definitions and their consequences we refer the interested reader to the complete version of this paper or to [17]. The main result of this paper is

THEOREM 3. *Let L be a Euclidean Functional that satisfies condition (A4) for two different values $m = m_1$, $m = m_2$ such that $gcd(m_1, m_2) = 1$, and that also satisfies condition (A5). Then $\exists c \ge 0$ such that $\frac{L(S_n)}{n^{(d-d')/d}} \to c$ in both expectation and probability, i.e.,*

$$\mathrm{E}\left(\frac{L(S_n)}{n^{(d-d')/d}}\right) \to c,$$

and

$$\forall \epsilon > 0, \quad \Pr\left(\left|\frac{L(S_n)}{n^{(d-d')/d}} - c\right| > \epsilon\right) \to 0.$$

Furthermore

$$\mathrm{VAR}\left(\frac{L(S_n)}{n^{(d-d')/d}}\right) \to 0.$$

Proof In what follows we set

$$H(n) = \frac{L(S_n)}{n^{(d-d')/d}}, \qquad E\psi_n = \mathrm{E}(L(S_n)),$$

$$\phi_n = \frac{\psi_n}{n^{(d-d')/d}} = \mathrm{E}(H(n)), \quad \tau_n = \mathrm{E}\left((L(S_n))^2\right),$$

and

$$\varphi_n = \frac{\tau_n}{n^{2(d-d')/d}} = \mathrm{E}\left(H^2(n)\right).$$

We first prove convergence in expectation by showing that ϕ_n is a binomially bounded sequence satisfying the conditions of Theorem 2. We then prove convergence in probability by demonstrating that $\mathrm{VAR}(H(n)) = (\varphi_n - \phi_n^2) \to 0$ and applying Chebyshev's inequality.

Let $m = m_1$. To prove that ϕ_n is a binomially bounded sequence with parameter $a = m^d$ let $S = S_n$ in (A3) and divide equation (3.6) by $n^{(d-d')/d}$. Then take expectations to find that

$$\frac{\psi_n}{n^{(d-d')/d}} = \frac{m^{d-d'}}{n^{(d-d')/d}}\mathrm{E}(\psi_X) + O\left(\frac{1}{\ln^2 n\sqrt{n}}\right)$$

where X is a binomially distributed random variable with parameters n and $p = m^{-d}$; to derive this equation we used conditions (A1) and (A2) to show that $\mathrm{E}(L(S_n \cap Q_i)) = \frac{1}{m^{d'}}\mathrm{E}(L(S_X))$. Standard Chernoff bounding techniques [6] show that $\frac{np}{X} = 1 + O\left(\frac{1}{\ln^2 n}\right)$ with probability $1 - n^{-\Omega(\ln n)}$. Therefore

$$\begin{aligned}\phi_n &= \mathrm{E}\left(\frac{\psi_X}{X^{(d-d')/d}}\right)\left[1 + O\left(\frac{1}{\ln^2 n}\right)\right] \\ &\quad + O\left(\frac{1}{\ln^2 n}\right) \\ &= \mathrm{E}(\phi_n)\left[1 + O\left(\frac{1}{\ln^2 n}\right)\right] + O\left(\frac{1}{\ln^2 n}\right)\end{aligned}$$

so ϕ_n is a binomially bounded sequence with parameter $a = m_1^d$. A similar argument shows that it is also a binomially bounded sequence with parameter $b = m_2^d$. Because $gcd(m_1, m_2) = 1$ we also have $\log_a b$ is irrational. Finally, dividing equations (3.7) and (3.8) by $n^{(d-d')/d}$ shows that at least one of condition (A) or (B) in the statement of Theorem 2 is satisfied. Theorem 2 therefore implies the existence of $c \ge 0$ such that $\lim_{n\to\infty} \phi_n = c$, i.e., $\mathrm{E}\left(\frac{L(S_n)}{n^{(d-d')/d}}\right) \to c$.

To prove that $\mathrm{VAR}(H(n)) \to 0$ refer back to condition (A3) setting $m = m_1$ and $S = S_n$. Taking expectations and using symmetry gives

$$\begin{aligned}\tau_n &= \frac{m^d}{m^{2d'}}\mathrm{E}(\tau_X) + \frac{m^{2d} - m^d}{m^{2d'}}\mathrm{E}(\tau_{X_1}\tau_{X_2}) \\ &\quad + \mathrm{E}\left(\psi_X \frac{n^{(d-d')/d}}{\ln^2 n}\right) + O\left(\frac{n^{2(d-d')/d}}{\ln^2 n}\right)\end{aligned}$$

where X, $X_1 = |S \cap Q_1|$, $X_2 = |S \cap Q_2|$ are all binomial random variables with parameters n and $p = m^{-d}$. Using Chernoff bounding techniques to show that X_1 and X_2 are strongly concentrated around their mean along with the previously proven fact that $\mathrm{E}(\phi_n) = O(1)$ we find that

$$\begin{aligned}\varphi_n &\le \frac{1}{m^d}\mathrm{E}(\varphi_X) \\ &\quad + \left(1 - \frac{1}{m^d}\right)\mathrm{E}(\psi_X^2)\left[1 + O\left(\frac{1}{\ln^2 n}\right)\right] \\ &\quad + O\left(\frac{1}{\ln^2 n}\right).\end{aligned} \tag{3.9}$$

Jensen's inequality [5, p. 161] tells us that $\psi_n^2 \le \varphi_n$ for all n, so

$$\varphi_n \le \mathrm{E}(\varphi_X)O\left(\frac{1}{\ln^2 n}\right) + O\left(\frac{1}{\ln^2 n}\right),$$

φ_n is a binomially bounded sequence and Lemma 3.1 therefore says that it is a bounded sequence. This in turn implies the existence of $c' < \infty$ such that $\lim_{m\to\infty}\sup_{n>m}\varphi_n = c'$ while Jensen's inequality tells us that $c^2 \le c'$. We will now see that $c' \le c^2$, proving that $\lim_{n\to\infty}\varphi_n = c^2$.

Suppose, by contradiction, that $c' = c^2+\delta$ for some $\delta > 0$. Fix $\epsilon > 0$. For all large enough n, $\mathrm{E}(\psi_n^2) < c^2+\epsilon$ and $\mathrm{E}(\tau_n) < c^2+\delta+\epsilon$. Plugging back into equation (3.9) this implies that for all large enough n

$$\begin{aligned}\varphi_n &\le \frac{1}{m^d}(c^2+\delta+\epsilon) + \left(1-\frac{1}{m^d}\right)(c^2+\epsilon) \\ &\quad + O\left(\frac{1}{\ln^2 n}\right) \\ &\le c^2 + \frac{\delta}{m^d} + \epsilon + O\left(\frac{1}{\ln^2 n}\right)\end{aligned}$$

contradicting the definition of δ. Thus $c' = c^2$ and

$$\lim_{n\to\infty}\mathrm{VAR}(H(n)) = \lim_{n\to\infty}(\varphi_n - \phi_n^2) = 0.$$

For any fixed $\epsilon > 0$ and large enough n, $|\phi_n - c| \le \epsilon/2$. Thus, for large enough n Chebshev's inequality yields

$$\begin{aligned}\Pr(|H(n)-c| > \epsilon) &\le \Pr(|H(n)-\phi_n| \ge \epsilon/2) \\ &\le 4\frac{\mathrm{VAR}(H(n))}{\epsilon^2}.\end{aligned}$$

Since $\lim_{n\to\infty}\mathrm{VAR}(H(n)) = 0$ this implies convergence in probability. □

To conclude this subsection note that Lemma 2.1 shows that the MWT functional satisfies all of the conditions of Theorem 3, thus proving Theorem 1 as promised.

3.3 Other Applications Theorem 3 is quite general and permits the analysis of a variety of other problems. To illustrate its use we describe how it proves convergence for a generalization of the MWT. Let $S \subset \Re^d$. We define a "triangulation" of S to be a simplicial complex in $\Re^d$ whose vertices are exactly the points in S. Suppose $\mathcal{T}$ is some triangulation of S. Define $w_{d'}(\mathcal{T})$, the d'-dimensional weight of $\mathcal{T}$, to be the sum of the d'-dimensional volumes of all d'-dimensional flats in $\mathcal{T}$. For exanple, if $d = 3$ then $\mathcal{T}$ partitions the convex hull of S into tetrahedrons, $w_1(\mathcal{T})$ is the sum of the lengths of the edges of $\mathcal{T}$ and $w_2(\mathcal{T})$ is the sum of the areas of all of the triangles that are faces in $\mathcal{T}$. Finally set $MWT_{d'}^d(S)$ to be the triangulation of S with minimal $w_{d'}()$ weight. For example MWT_1^2 is exactly the $MWT(S)$ we have previously analyzed.

It is now quite straightforward to use Theorem 3 to generalize Theorem 1 as follows (proof omitted in this extended abstract):

THEOREM 4. *Let $x_1, x_2, \ldots$ be points chosen independently from the uniform distribution over the unit d-cube $[0,1]^d$ and set $S = \{x_1,\ldots,x_n\}$. Then, if $d' = d-1$ or $d' < d/2$ there exists a constant $c_{d'}^d > 0$ such that $\frac{\mathrm{MWT}_{d'}^d(S_n)}{n^{(d-d')/d}}$ converges to $c_{d'}^d$ in both expectation and probability, i.e.,*

$$\mathrm{E}\left(\frac{\mathrm{MWT}_{d'}^d(S_n)}{n^{(d-d')/d}}\right) \to c_{d'}^d,$$

and,

$$\forall\epsilon > 0, \quad \Pr\left(\left|\frac{\mathrm{MWT}_{d'}^d(S_n)}{n^{(d-d')/d}} - c_{d'}^d\right| > \epsilon\right) \to 0.$$

As another application we offer a theorem concerning the degrees of vertices in the Delaunay triangulations. of random points. Given $S \subset \Re^2$ let $Del(S)$ be the Delaunay triangulation of S. we can prove

THEOREM 5. *Let $x_1, x_2, \ldots$ be points chosen independently from the uniform distribution over the unit square $[0,1]^2$ and set $S = \{x_1,\ldots,x_n\}$. Let $d_n^{(i)}$ to be the probability that a random point in S_n has degree i in $Del(S_n)$. Then there exist constants $d^{(i)} > 0$ for $i \ge 2$*

$$\lim_{n\to\infty} d_n^{(i)} = d^{(i)}$$

We omit the full proof in this extended abstract (it will be given elsewhere) but sketch the idea behind it. Consider the functional

$$D^{(i)}(S) = |\{p \in S : p \text{ has degree } i \text{ in } Del(S)\}|.$$

By definition $d_n^{(i)} = \mathrm{E}\left(\frac{D^{(i)}(S_n)}{n}\right)$. The functional can be shown to satisfy Theorem 3 with scaling factor $d' = 0$ and thus there exists a constant $d^{(i)}$ such that

$$\lim_{n\to\infty} d_n^{(i)} = \lim_{n\to\infty}\mathrm{E}\left(\frac{D^{(i)}(S_n)}{n}\right) = d^{(i)}.$$

We note that a similar theorem has previously been proven analyzing the degrees of points in the minimal spanning trees of random point sets [16] but the techniques used there do not seem applicable here.

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