Q1.
(a) Since $\varepsilon = 0.2$, $s = 0.4$ and $N = 10$, we get $\varepsilon N = 2$ and $(s - \varepsilon)N = 2$.

For Algorithm 1, it does not have an $\varepsilon$-deficient synopsis.
- The item whose true frequency less than $(s - \varepsilon)N$ (i.e. $I_4$) is **NOT** classified as an infrequent item in the algorithm output.

For Algorithm 2, it has an $\varepsilon$-deficient synopsis.
- It has no false negative.
- The difference between the estimated frequency and the true frequency of each item $\leq \varepsilon N$ (i.e. the differences for $I_1, I_3, I_3$ and $I_4$ are 2,1,1,0).
- All items whose true frequencies less than $(s - \varepsilon)N$ (i.e. $I_4$) are classified as infrequent items in the algorithm output.

For Algorithm 3, it has an $\varepsilon$-deficient synopsis.
- It also introduces no false negative.
- The difference between the estimated frequency and the true frequency of each item $\leq \varepsilon N$ (i.e. the differences for $I_1, I_2, I_3$ and $I_4$ are 1,1,1,0).
- All items whose true frequencies less than $(s - \varepsilon)N$ (i.e. $I_4$) are classified as infrequent items in the algorithm output.

(b) (i) memory $= \sum_{i=1}^{m} \left( \frac{m}{i} \right) \frac{1}{\varepsilon} (\log \varepsilon N)$

(OR $2^M - 1 \frac{1}{\varepsilon} (\log \varepsilon N)$ where $M$ is the total number of items.).

(ii) It is not nicely bounded because the memory consumption contains an exponential factor on $m$ or $M$.  

Q2
(a)

Let $N$ be the total number of items read so far.

The first $2t$ elements find their way into $S$.

When the sampling rate is $r \geq 2$,
we have $N = rt + \Delta$
where $\Delta \in [1, rt)$

Consider

\[ N = rt + \Delta \]

\[ \frac{1}{r} = \frac{t + \Delta}{N} \]

\[ \frac{1}{r} \geq \frac{t}{N} \]

Consider an element $e$ in $S$

The stored frequency $\tilde{f}$ of $e$ is at most the true frequency $f_0$ of $e$

Why is the value of $\tilde{f}$ different from $f_0$? Why not the same?
This is because the stored frequency $\tilde{f}$ is deducted by tossing a coin repeatedly. If the outcome is “tail”,
the stored frequency will be decremented.

The reason why $e$ still remains in $S$ is that the outcome of the last toss is “head”.

Note that

\[ P(\text{Head}) = \frac{1}{r} \]

and \[ P(\text{Tail}) = 1 - \frac{1}{r} \]

Structure of the proof:
A. First, we prove the following

with confidence at least $1-\delta$,

\[ f_0 - \tilde{f} \leq \varepsilon N \]
B. Second, with the above result, we prove the following

The algorithm returns the $\varepsilon$-deficient synopsis.

Part A

Lemma 1

Let $C$ be a non-negative integer.

$$\operatorname{Prob}(f_0 - \tilde{f} = C) \leq \frac{1}{r} \left(1 - \frac{1}{r}\right)^C$$

where $r$ is the current sampling rate.

Proof:

Let $x$ be the total number of the changes of sampling rate.
Let $r_i$ be the sampling rate at the $i$-th time of changing sampling rate.

Consider the two scenarios.
The first scenario is that after each boundary, the entry for an item is still in the memory.

For the $i$-th time of changing sampling rate,
we decrement the stored frequency of $e$ by $n_i$ times
(i.e., there are $n_i$ tails when we toss a coin repeatedly)

Thus, for each $i$ in $[1, x]$, we know that $P(\text{tail}) = 1 - 1/r_i$

$$\leq 1 - 1/r_x$$
The second scenario is that after the i-th boundary, the entry for an item is not in the memory.

Thus, for each \( i \) in \( [1, x] \), we know that \( P(\text{Unsuccessful insertion}) = 1 - \frac{1}{ri} \leq 1 - \frac{1}{rx} \).

Just after the \( x \)-th time of changing sampling rate, 
\( f_o - \tilde{f} \) is equal to \( C \).

There are \( C \) times of frequency loss of \( f_o \).

Each frequency loss is either (1) we observe a tail when we toss a coin (when we reach the boundary) or (2) we encounter an unsuccessful insertion (between two adjacent boundaries).

Since \( P(\text{tail}) \leq 1 - \frac{1}{rx} \) and \( P(\text{Unsuccessful insertion}) \leq 1 - \frac{1}{rx} \),
\[ P(\text{a frequency loss of } f_o) \leq 1 - \frac{1}{rx} \]

Since there are \( C \) times of frequency loss of \( f_o \) and we know that the outcome of the last coin toss is head (since the entry for the item is still in the memory),
\[ P(\text{there are } C \text{ times of frequency loss of } f_o) \leq \frac{1}{r_x} \left(1 - \frac{1}{r_x}\right)^C \]
**Lemma 2**

\[
\text{Prob} \left( f_0 - \tilde{f} \geq \varepsilon N \right) \leq e^{-\varepsilon t}
\]

**Proof:**

We want to calculate the probability that the difference between \( f_0 \) and \( \tilde{f} \) is at least \( \varepsilon N \) (i.e., \( \text{Prob} \left( f_0 - \tilde{f} \geq \varepsilon N \right) \)).

\[
\text{Prob} \left( f_0 - \tilde{f} \geq \varepsilon N \right)
= \text{Prob} \left( f_0 - \tilde{f} = \varepsilon N \right) + \text{Prob} \left( f_0 - \tilde{f} = \varepsilon N + 1 \right) + \text{Prob} \left( f_0 - \tilde{f} = \varepsilon N + 2 \right) + \ldots
\leq \frac{1}{r} \left( 1 - \frac{1}{r} \right)^{\varepsilon N} + \frac{1}{r} \left( 1 - \frac{1}{r} \right)^{\varepsilon N + 1} + \frac{1}{r} \left( 1 - \frac{1}{r} \right)^{\varepsilon N + 2} + \ldots
= \frac{1}{r} \left[ (1 - \frac{1}{r})^{\varepsilon N} + (1 - \frac{1}{r})^{\varepsilon N + 1} + (1 - \frac{1}{r})^{\varepsilon N + 2} + \ldots \right]
= \frac{1}{r} \times \frac{(1 - \frac{1}{r})^{\varepsilon N}}{1 - (1 - \frac{1}{r})}
= (1 - \frac{1}{r})^{\varepsilon N}
\leq (1 - \frac{t}{N})^{\varepsilon N} \quad \text{(because } \frac{1}{r} \geq \frac{t}{N} \text{)}
\leq e^{-\varepsilon t} \quad \text{(because } \lim_{N \to \infty} (1 + \frac{t}{N})^N = e^t \text{)}

**Lemma 3**

The probability that at least one of the frequent elements has \( f_0 - \tilde{f} \geq \varepsilon N \) is at most \( \delta \)

**Proof:**

The greatest no. of frequent elements (i.e., elements with frequency at least \( sN \)) is equal to

\[
= \frac{N}{sN}
= \frac{1}{s}
\]

Thus, the probability that at least one of the frequent elements has \( f_0 - \tilde{f} \geq \varepsilon N \) is
Lemma 4

With confidence at least 1-δ,
\[ f_0 - \tilde{f} \leq \varepsilon N \] \hspace{1cm} (1)

Proof:

The probability that each frequent element has \( f_0 - \tilde{f} \leq \varepsilon N \)

\[ = 1 - \text{probability that at least one of the frequent elements has } f_0 - \tilde{f} \geq \varepsilon N \]

(the boundary case (i.e., when \( f_0 - \tilde{f} = \varepsilon N \)) is not handled here for the sake of simplicity)

\[ \geq 1 - \delta \]

In other words,

\[ \text{Prob} \left( f_0 - \tilde{f} \leq \varepsilon N \right) \geq 1 - \delta \]

We can also write

with confidence at least 1-δ,

\[ f_0 - \tilde{f} \leq \varepsilon N \] \hspace{1cm} (1)

Lemma 4 is the result for Part A.

Part B

Next, we check the three conditions of the \( \varepsilon \)-deficient synopsis.

The second condition is satisfied with confidence at least 1-δ.

Consider the first condition.

Consider an item with true frequency \( f_0 \geq sN \)

From (1),
\[ \tilde{f} + \varepsilon N \geq f_0 \]
\[ \Rightarrow \tilde{f} + \varepsilon N \geq sN \]

We conclude that
an item has true frequency \( f_0 \geq sN \)

\[ \Rightarrow \text{it has } \tilde{f} + \varepsilon N \geq sN \text{ (during the execution of the algorithm)} \]
\[ \Rightarrow \text{this item is in the algorithm output} \]

Thus, the first condition is satisfied.

Consider the third condition.

Consider an item with true frequency \( f_0 \leq (s-\varepsilon)N \)

Thus, we have \( f_0 \leq sN-\varepsilon N \)
\[ f_0 + \varepsilon N \leq sN \]
\[ \tilde{f} + \varepsilon N \leq sN \] (since \( \tilde{f} \leq f_0 \))

We conclude that this item is NOT in the algorithm output (i.e., it is classified as an infrequent item).

Thus, the third condition is satisfied.

(b) Differences:
1. Sticky Sampling Algorithm is a probabilistic algorithm.
   Lossy Counting Algorithm is a deterministic algorithm.
2. Storage of Sticky Sampling Algorithm is independent of \( N \)
   Storage of Lossy Counting Algorithm is dependent on \( N \)

(c) Differences:
1. Lossy Counting Algorithm requires an error parameter \( \varepsilon \) which we can deduce the memory storage
   Space-saving Algorithm requires the memory parameter \( M \) which we can deduce the error
2. Given the same memory (i.e., the max. memory that we can use), Space-Saving Algorithm has smaller error than Lossy Counting Algorithm.

Q3
(a)(i)
Let \( M \) be the memory size (in bytes (not the number of entries in form of \( (e, f, \Delta) \) discussed in class)). We divide the memory into two parts:
1. Part 1: buffer which stores the incoming data
2. Part 2: summary of previous batches.
We allocate the memory for part 1 with \( B \) bytes. And the memory for part 2 is \( M-B \) bytes.
Since B is given, the memory size for Part 1 and the memory size for part 2 are fixed. We divide the memory for Part 2 into 4 partitions of equal size (We do not divide the memory into 4 partitions of adjustable size because in the worst case, in order to minimize the error, we also need the equal-sized partitions.) Thus, we have four batches. Each batch is associated with a summary. For each batch, we can update the summary (associated to the batch) like what the original space-saving algorithm does.

Given a set \( S \) of data points, the summary stored in the Space-Saving algorithm for this set is denoted by \( \text{SpaceSaving}(S) \). Let \( X = \text{SpaceSaving}(S) \). \( X \) contains two components. The first component denoted by \( X.E \) contains a list of entries each in form of \( (e, f, \Delta) \) where \( e \) is the item number, \( f \) is the frequency of the item (recorded after this entry is created) and \( \Delta \) is the maximum possible error in \( f \). The second component denoted by \( X.p \) is equal to the variable \( p_e \) used in algorithm Space-Saving discussed in class (i.e., the greatest possible frequent error).

Output:

Let \( f(e, B_i) \) be the count of element \( e \) stored in the summary of batch \( B_i \). If the entry \( (e, f, \Delta) \) for \( e \) exists in the summary of batch \( B_i \), \( f(e, B_i) \) is equal to \( f \). Otherwise, it is equal to 0.

Let \( \Delta(e, B_i) \) be the \( \Delta \) value of element \( e \) stored in batch \( B_i \). If the entry \( (e, f, \Delta) \) for \( e \) exists in the summary \( X \) of batch \( B_i \), \( \Delta(e, B_i) \) is equal to \( \Delta \). Otherwise, it is equal to \( X.p \).

For each item \( e \), calculate \( f_e = \sum_{i=1}^{4} f(e, B_i) \) and \( \Delta_e = \sum_{i=1}^{4} \Delta(e, B_i) \)

Get a list of items where \( f_e + \Delta_e \geq sN \) where \( N=4B \)

(ii) The memory for part 2 is \( M-B \). Since the memory for part 2 is divided into 4 equal-sized parts, each part occupies \( (M-B)/4 \).

Consider a part containing a summary \( X \). Assume that the second component of \( X \) (i.e., \( X.p \)) occupies 1 byte. Since the second component of \( X \) occupies 1 byte, the first component of \( X \) occupies \( (M-B)/4 - 1 \).

Let \( m \) be the size occupied by a single entry in form of \( (e, f, \Delta) \). The total number of entries each in form of \( (e, f, \Delta) \) is equal to \( [(M-B)/4 - 1]/m \).

For each batch, the greatest error in fraction = \( \frac{1}{M-B} - \frac{4}{M-B-4} \) = \( \frac{4m}{M-B-4} \).

The overall greatest error in count = \( \frac{4m}{M-B-4} \times 4B = \frac{16mB}{M-B-4} \)

The overall greatest error in fraction = \( \frac{16mB}{M-B-4} \times \frac{1}{4B} = \frac{4m}{M-B-4} \)
(b)(i) The answer of (b)(i) is the same as the answer of (a)(i). But we need to update the output part. Note that the incomplete batch contains all items over data streams instead of the summary. Let \( f(e, B_i) \) be the count of element \( e \) stored in batch \( B_i \). The definition is the same as (a)(i).

Let \( \Delta(e, B_i) \) be the \( \Delta \) value of element \( e \) stored in batch \( B_i \). The definition is the same as (a)(i).

Let \( I \) be the incomplete batch. Let \( f(e, I) \) be the exact count of element \( e \) in the incomplete batch \( I \).

Consider batch \( B_1 \), for each item \( e \), calculate \( Y_e = \min\{ f(e, B_1) + \Delta(e, B_1), 0.5B \} \)

Consider batch \( B_2, B_3, B_4, I \),

For each item \( e \), calculate \( f_e = \sum_{i=2}^{4} f(e, B_i) + f(e, I) \) and \( \Delta e = \sum_{i=2}^{4} \Delta(e, B_i) + 0 \)

Get a list of items where \( f_e + \Delta_e + Y_e \geq sN \) where \( N = 4B \).

(ii) For each of batches \( B_2, B_3, B_4 \),

the greatest error in fraction = \( \frac{1}{M - B - 1} = \frac{4m}{M - B - 4} \).

For batch \( I \), the greatest error = 0 in both count and fraction.

For batch \( B_1 \), the greatest error in fraction is 0.5, so the greatest error in count is 0.5B.

The overall greatest error in count = \( \frac{4m}{M - B - 4} \times 3B + 0 + 0.5B = \frac{12mB}{M - B - 4} + 0.5B \)

So, the overall greatest error in fraction = \( \frac{1}{4B} \times \left( \frac{12mB}{M - B - 4} + 0.5B \right) = \frac{3m}{M - B - 4} + \frac{1}{8} \).
Q4.

Stochastic matrix:

\[
\begin{pmatrix}
A & B & C & D \\
0 & 1 & 0 & 0 \\
0.5 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Equation to be solved:

\[
\begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix} = 0.8 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}
\]

So the ranking is C, D, A, B.
Q5.
(a) Skyline = \{a, c\}
(b) (i) Subspace \{D_1, D_2\}, Skyline = \{c, e\}
Subspace \{D_2, D_3\}, Skyline = \{a, d\}
Subspace \{D_1, D_3\}, Skyline = \{a, c\}
(ii) No. For example, a is a skyline point w.r.t. \(D\), but not a skyline point w.r.t. \{D_1, D_2\}
(iii) No. For example, e is a skyline point w.r.t. \{D_1, D_2\}, but not a skyline point w.r.t. \(D\).
(iv) We prove it by contradiction.
Suppose there does not exist a point \(p\) in \(S'\) such that \(p\) is in \(S\). That is, \(\forall p \in S' \Rightarrow p \notin S\).
\[\therefore p \notin S\]
\[
\therefore \exists a \text{ point } q \in S \text{ which dominates } p \text{ w.r.t. } D.
\]
i.e. \(\forall D_i \in D, q.D_i \leq p.D_i \) and \(\exists D'_i \in D, q.D'_i < p.D'_i\)
Consider two cases:
(1) \(D'\) contains \(D'_i\) (where \(D'_i\) is a dimension where \(q.D'_i < p.D'_i\))
\[\Rightarrow \forall D_i \in D', q.D_i \leq p.D_i \text{ and } \exists D'_i \in D', q.D'_i < p.D'_i\]
\[\Rightarrow q \text{ dominates } p \text{ w.r.t. } D'\]
\[\Rightarrow p \notin S'\]
It is a contradiction.
(2) \(D'\) does not contain any \(D'_i\) (where \(D'_i\) is a dimension where \(q.D'_i < p.D'_i\))
\[\Rightarrow \forall D_i \in D', q.D_i \leq p.D_i\]
\[\therefore p \in S'\]
\[\therefore \exists a \text{ point q in S' which dominates p w.r.t. D'}.\]
We deduce that \(\forall D_i \in D', q.D_i = p.D_i\).
\[\Rightarrow q \in S'\]
\[\therefore q \in S\]
\[\therefore \text{ We conclude that there exists a point q in S' such that q is in S}\.\]
It is also a contradiction.
Q6.
(a) Yes.
It is sufficient to prove “SKY(P_{a<b}) ⊆ SKY(P_a)” (because X ⊆ SKY(P_{a<b})).
Note that P_a ⊆ P_{a<b}.
It is easy to verify that for any two sets P and P’ containing preferences, if P ⊆ P’, then SKY(P’) ⊆ SKY(P).
Since there are more dominance relationships in the data set for P’ and thus more points in the dataset are dominated.

Since P_a ⊆ P_{a<b}, we have SKY(P_{a<b}) ⊆ SKY(P_a).
Thus, X ⊆ SKY(P_a).

(b) No.
We give a counter-example as follows.
h_1: (100, Royal)
h_2: (200, Marriott)
h_3: (250, Marriott)
Let a = “Marriott” and b = “Royal”.
We have:
   SKY(P_a) = \{h_1, h_2\}
   SKY(P_b) = \{h_1\}
   SKY(P_{a<b}) = \{h_1, h_2\}
   SKY(P_a) ∩ SKY(P_b) = \{h_1\}
So, X = SKY(P_{a<b}) – [SKY(P_a) ∩ SKY(P_b)] = \{h_2\}.
Note that X ∉ SKY(P_b).

(c) Yes.
Note that the major difference between P_a ∪ P_b and P_{a<b} is that P_a ∪ P_b is equal to P_{a<b} but it contains one more pair (b, a).
Also note that SKY(P_a) ∩ SKY(P_b) corresponds to SKY(P_a ∪ P_b).
(Note that P_a ∪ P_b can contain a contradictory order (but it doesn’t matter).)

In (a), we know that if P ⊆ P’, then SKY(P’) ⊆ SKY(P). We deduce that for each point q ∈ SKY(P) – SKY(P’), there exists a pair (x, y) ∈ P’-P such that the C value of q is y.

In our case, where P’ = P_a ∩ P_b and P = P_{a<b}, we conclude that for each point q ∈ SKY(P_{a<b}) – SKY(P_a ∪ P_b), we have the pair (b, a) (∈ [P_a ∪ P_b – P_{a<b}]) such that the C value of q is a.

Note that X = SKY(P_{a<b}) – SKY(P_a ∪ P_b) (or SKY(P_{a<b}) – [SKY(P_a) ∩ SKY(P_b)]),
We conclude that each point in X has the C value equal to a.

From (a), we know that X ⊆ SKY(P_a).
We deduce that X ⊆ K(SKY(P_a), a).
(d) No.
We can use the counter example in (b).
$X = \{h_2\}$.
The C value of $h_2$ is a. Thus, $h_2 \not\in K(SKY(P_b), b)$. Thus, $X \not\subseteq K(SKY(P_b), b)$. 