Dynamic Indexability
and
Lower Bounds for Dynamic
One-Dimensional Range Query Indexes

Ke Yi
HKUST
First Annual SIGMOD Programming Contest (to be held at SIGMOD 2009)

- “Student teams from degree granting institutions are invited to compete in a programming contest to develop an indexing system for main memory data.”
- “The index must be capable of supporting range queries and exact match queries as well as updates, inserts, and deletes.”
- “The choice of data structures (e.g., B-tree, AVL-tree, etc.) ... is up to you.”
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- We think these problems are so basic that every DB grad student should know, but do we really have the answer?
Answer: Hash Table and B-tree!

- Indeed, (external) hash tables and B-trees are both fundamental index structures that are used in all database systems.
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External memory model (I/O model):

- Memory of size $m$
- Each I/O reads/writes a block
- Disk partitioned into blocks of size $b$
The B-tree
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A range query in $O(\log_b n + k/b)$ I/Os

$k$: output size
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$k$: output size

$log_b n - log_b m = log_b \frac{n}{m}$
The height of B-tree never goes beyond 5 (e.g., if $b = 100$, then a B-tree with 5 levels stores $n = 10$ billion records). We will assume $\log_b \frac{n}{m} = O(1)$. 

$\log_b n - \log_b m = \log_b \frac{n}{m}$

$\log_b n + k/b$ I/Os

$k$: output size

A range query in $O(\log_b n + k/b)$ I/Os
Now Let’s Go Dynamic

- Focus on insertions first: Both the B-tree and hash table do a search first, then insert into the appropriate block
- B-tree: Split blocks when necessary
- Hashing: Rebuild the hash table when too full; *extensible hashing* [Fagin, Nievergelt, Pippenger, Strong, 79]; *linear hashing* [Litwin, 80]
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- Cannot hope for lower than 1 I/O per insertion only if the changes must be committed to disk right away (necessary?)
  - Otherwise we probably can lower the amortized insertion cost by buffering, like numerous problems in external memory, e.g. stack, priority queue,... All of them support an insertion in $O(1/b)$ I/Os — the best possible
Dynamic B-trees for Fast Insertions

- **LSM-tree** [O’Neil, Cheng, Gawlick, O’Neil, 96]: Logarithmic method + B-tree
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  - Query: $O\left(\log_{\ell} \frac{n}{m} + \frac{k}{b}\right)$

![Diagram of memory layout with notations: $m$, $\ell m$, $\ell^2 m$]
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Usually $\ell$ is set to be a constant, then they both have $O\left(\frac{1}{b} \log \frac{n}{m}\right)$ insertion and $O\left(\log \frac{n}{m} + \frac{k}{b}\right)$ query
More Dynamic B-trees

- *Buffer tree* [Arge, 95]
- *Yet another B-tree* (Y-tree) [Jermaine, Datta, Omiecinski, 99]
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  - Query: \(O\left(\log \frac{n}{m} + \frac{k}{b}\right)\), much worse than the static B-tree’s \(O(1+\frac{k}{b})\); if \(O(1+\frac{k}{b})\) query required, insertion cost becomes \(O\left(\frac{b^\epsilon}{b}\right)\)
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- No further development in the last 10 years. So, seems we can’t do better, can we?
Main Result

For any dynamic range query index with a query cost of $q + O(k/b)$ and an amortized insertion cost of $u/b$, the following tradeoff holds

$$\begin{align*}
q \cdot \log(u/q) &= \Omega(\log b), & \text{for } q < \alpha \ln b, \alpha \text{ is any constant;} \\
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Current upper bounds:

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<thead>
<tr>
<th>( q )</th>
<th>( u )</th>
</tr>
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<tbody>
<tr>
<td>( \log \frac{n}{m} )</td>
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</tr>
<tr>
<td>1</td>
<td>( \left( \frac{n}{m} \right)^{\epsilon} )</td>
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The technique of [Brodal, Fagerberg, 03] for the predecessor problem can be used to derive a tradeoff of

$$q \cdot \log (u \log^2 \frac{n}{m}) = \Omega(\log \frac{n}{m}).$$
Lower Bound Model: Dynamic Indexability

- *Indexability*: [Hellerstein, Koutsoupias, Papadimitriou, 97]
Lower Bound Model: Dynamic Indexability

- **Indexability**: [Hellerstein, Koutsoupias, Papadimitriou, 97]

- Objects are stored in disk blocks of size up to $b$, possibly with redundancy.

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  a query reports \{2,3,4,5\}

  | 4 7 9 | 1 2 4 | 3 5 8 | 2 6 7 | 1 8 9 | 4 5 |

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  cost = 2

  \[
  \begin{array}{ccccccc}
  4 & 7 & 9 & 1 & 2 & 4 & 3 & 5 & 8 \\
  2 & 6 & 7 & 1 & 8 & 9 & 4 & 5 \\
  \end{array}
  \]

- Objects are stored in disk blocks of size up to $b$, possibly with redundancy.
  Redundancy $r = \frac{\text{total \# blocks}}{\lceil \frac{n}{b} \rceil}$

- The query cost is the minimum number of blocks that can cover all the required results (search time ignored!).
  Access overhead $A = \frac{(\text{worst-case}) \text{ query cost}}{\lceil \frac{k}{b} \rceil}$
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- Similar in spirit to popular lower bound models: cell probe model, semigroup model
Previous Results on Indexability

- Nearly all external indexing lower bounds are under this model
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- 2D range queries: \( r = \Omega\left(\frac{\log(n/b)}{\log A}\right) \) [Hellerstein, Koutsoupias, Papadimitriou, 97], [Koutsoupias, Taylor, 98], [Arge, Samoladas, Vitter, 99]
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- 2D stabbing queries: $A_0 A_1^2 = \Omega \left( \frac{\log(n/b)}{\log r} \right)$ [Arge, Samoladas, Yi, 04]
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  - Adding dynamization makes it much more interesting!
Dynamic Indexability

- Still consider only insertions
Dynamic Indexability

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memory of size $m$

| time $t$: 1 2 7 |
| blocks of size $b = 3$ |
| 4 7 9 | 4 5 |

← snapshot
Dynamic Indexability

- Still consider only insertions

memory of size $m$

| time $t$: | 1 2 7 |
| blocks of size $b = 3$ | 4 7 9 | 4 5 |

| time $t + 1$: | 1 2 6 7 |
| blocks of size $b = 3$ | 4 7 9 | 4 5 |

← snapshot

6 inserted
Dynamic Indexability

- Still consider only insertions

<table>
<thead>
<tr>
<th>Time</th>
<th>Memory of size $m$</th>
<th>Blocks of size $b = 3$</th>
<th>Blocks Inserted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>1 2 7</td>
<td>4 7 9 4 5</td>
<td></td>
</tr>
<tr>
<td>$t+1$</td>
<td>1 2 6 7</td>
<td>4 7 9 4 5</td>
<td>6 inserted</td>
</tr>
<tr>
<td>$t+2$</td>
<td></td>
<td>4 7 9 1 2 5 6 8</td>
<td>8 inserted</td>
</tr>
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← snapshot
Dynamic Indexability

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- Redundancy (access overhead) is the worst redundancy (access overhead) of all snapshots

$\leftarrow$ snapshot

6 inserted

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- Redundancy (access overhead) is the worst redundancy (access overhead) of all snapshots

- Update cost: $u = \text{the average transition cost per } b \text{ insertions}$
Main Result Obtained in Dynamic Indexability

**Theorem:** For any dynamic 1D range query index with access overhead $A$ and update cost $u$, the following tradeoff holds, provided $n \geq 2mb^2$:

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\begin{align*}
A \cdot \log(u/A) &= \Omega(\log b), & \text{for } A < \alpha \ln b, \alpha \text{ is any constant;} \\
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Because a query cost $O(q + \lceil k/b \rceil)$ implies $O(q \cdot \lceil k/b \rceil)$

The lower bound doesn’t depend on the redundancy $r$!
The Ball-Shuffling Problem

\[ b \text{ balls} \rightarrow A \text{ bins} \]
The Ball-Shuffling Problem

$b$ balls $\rightarrow$ $A$ bins

$\Rightarrow$ cost $= 1$
The Ball-Shuffling Problem

\( b \) balls \quad \rightarrow \quad A \) bins

\[ \begin{align*}
\bullet \bullet \bullet \bullet \bullet \bullet \bullet & \quad \rightarrow \quad \begin{array}{cc}
\vdots & \vdots \\
\end{array} \\
\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet & \quad \rightarrow \quad \begin{array}{cc}
\vdots & \vdots \\
\bullet & \vdots \\
\end{array} \\
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\vdots & \vdots \\
\bullet & \bullet \\
\end{array}
\end{align*} \]

cost = 1

cost = 2

cost of putting the ball directly into a bin = \# balls in the bin + 1
The Ball-Shuffling Problem

\( b \) balls \quad \rightarrow \quad A \text{ bins}
The Ball-Shuffling Problem

\( b \) balls → \( A \) bins

Shuffle: →

cost = 5
The Ball-Shuffling Problem

$b$ balls

$\rightarrow$

$A$ bins

Shuffle:

Cost of shuffling = $\#$ balls in the involved bins

cost = 5
The Ball-Shuffling Problem

\[ b \text{ balls} \rightarrow \begin{array}{c}
\begin{array}{c}
\vspace{1cm}
\vspace{1cm}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\vspace{1cm}
\vspace{1cm}
\end{array}
\end{array} \]

\[ \begin{array}{c}
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\vspace{1cm}
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\end{array}
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\[ \text{Shuffle:} \rightarrow \begin{array}{c}
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Cost of shuffling = \# balls in the involved bins

Putting a ball directly into a bin is a special shuffle

\[ \text{cost} = 5 \]
The Ball-Shuffling Problem

$b$ balls $\rightarrow$ $A$ bins

Cost of shuffling = \# balls in the involved bins

Putting a ball directly into a bin is a special shuffle

Goal: Accommodating all $b$ balls using $A$ bins with minimum cost
Theorem: The cost of any solution for the ball-shuffling problem is at least

\[ \Omega(A \cdot b^{1+\Omega(1/A)}), \quad \text{for } A < \alpha \ln b \text{ where } \alpha \text{ is any constant}; \]

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cost lower bound
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cost lower bound
Theorem: The cost of any solution for the ball-shuffling problem is at least

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Cost lower bound

Tight (ignoring constants in big-Omega) for \( A = O(\log b) \) and \( A = \Omega(\log^{1+\epsilon} b) \)
The Workload Construction

round 1: ● ● ● ● ● ⋯

time → keys
The Workload Construction

round 1:  ●  ●  ●  ●  ●  ●  ●

round 2:  ●  ●  ●  ●  ●  ●  ●

time

keys
The Workload Construction

round 1: ● ● ● ● ● ● ● ●
round 2: ● ● ● ● ● ● ● ●
round 3: ● ● ● ● ● ● ● ●
...       
round b: ● ● ● ● ● ● ● ●

keys

time
The Workload Construction

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<tr>
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<td>$# $ queries $\geq 2mb$</td>
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| round 2: | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ |    |
| round 3: | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ |    |
| ...     |    |    |    |    |    |    |    |    |
| round $b$: | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ | ⬤ |
The Workload Construction

Queries that we require the index to cover with $A$ blocks

$\# \text{ queries} \geq 2mb$

Snapshots of the dynamic index considered
There exists a query such that

- The $\leq b$ objects of the query reside in $\leq A$ blocks in all snapshots
- All of its objects are on disk in all $b$ snapshots (we have $\geq mb$ queries)
- The index moves its objects $ub$ times in total
The Reduction

An index with update cost $u$ and access overhead $A$ gives us a solution to the ball-shuffling game with cost $ub$ for $b$ balls and $A$ bins
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$$\downarrow$$

$$\begin{cases} A \cdot \log(u/A) = \Omega(\log b), & \text{for } A < \alpha \ln b, \alpha \text{ is any constant;} \\ u \cdot \log A = \Omega(\log b), & \text{for all } A. \end{cases}$$
Ball-Shuffling Lower Bound Proof

\[ \Omega(A \cdot b^{1+\Omega(1/A)}), \quad \text{for } A < \alpha \ln b \text{ where } \alpha \text{ is any constant}; \]
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Will show: Any algorithm that handles the balls with an average cost of \(u\) using \(A\) bins cannot accommodate \((2A)^{2u}\) balls or more.

\[b < (2A)^{2u}, \text{ or } u > \frac{\log b}{2 \log(2A)}, \text{ so the total cost of the algorithm is } ub = \Omega(b \log_A b).\]
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  - \( u = 1 \): Can handle at most \( A \) balls.
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\[ u \rightarrow u + \frac{1}{2}? \]
Need to show: Any algorithm that handles the balls with an average cost of \( u + \frac{1}{2} \) using \( A \) bins cannot accommodate \( (2A)^{2u+1} \) balls or more.

\[ \leq \]

To handle \( (2A)^{2u+1} \) balls, any algorithm has to pay an average cost of more than \( u + \frac{1}{2} \) per ball, or

\[ \left( u + \frac{1}{2} \right) (2A)^{2u+1} = (2Au + A)(2A)^{2u} \]

in total.
Ball-Shuffling Lower Bound Proof (2)

- Need to show: Any algorithm that handles the balls with an average cost of $u + \frac{1}{2}$ using $A$ bins cannot accommodate $(2A)^{2u+1}$ balls or more.

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- There are at most $A$ bad batches
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- Each good batch contributes at least $(2A)^{2u}$ to the “interference” cost
Lower Bound Proof: The Real Work

\[ \Omega(A \cdot b^{1+\Omega(1/A)}), \quad \text{for } A < \alpha \ln b \text{ where } \alpha \text{ is any constant; } \]
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- The merging lemma: There is an optimal ball-shuffling algorithm that only uses merging shuffles.

- Let \( f_A(b) \) be the minimum cost to accommodate \( b \) balls with \( A \) bins.

- The recurrence

\[
    f_{A+1}(b) \geq \min_{k, x_1 + \cdots + x_k = b} \left\{ f_A(x_1 - 1) + \cdots + f_A(x_k - 1) + kx_1 + (k - 1)x_2 + \cdots + x_k - b \right\}
\]
Open Problems and Conjectures

- 1D range reporting
  - Current lower bound: query $\Omega(\log b)$, update $\Omega(\frac{1}{b} \log b)$. Improve to $(\log \frac{n}{m}, \frac{1}{b} \log \frac{n}{m})$?
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- Closely related problems: range sum (partial sum), predecessor search
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B-tree + logarithmic method |
| predecessor           | $\Omega : \left( \log n, \log n \right)$  
$\Omega : \ldots$          | $\Omega : \ldots$            |
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$\Omega : \text{open}$  

Optimal for all three?
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THANK YOU

Q and A