

# The Space Complexity of 2-Dimensional Approximate Range Counting

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## Abstract

We study the problem of 2-dimensional orthogonal range counting with additive error. Given a set  $P$  of  $n$  points drawn from an  $n \times n$  grid and an error parameter  $\varepsilon$ , the goal is to build a data structure, such that for any orthogonal range  $R$ , the data structure can return the number of points in  $P \cap R$  with additive error  $\varepsilon n$ . A well-known solution for this problem is the  $\varepsilon$ -approximation. Informally speaking, an  $\varepsilon$ -approximation of  $P$  is a subset  $A \subseteq P$  that allows us to estimate the number of points in  $P \cap R$  by counting the number of points in  $A \cap R$ . It is known that an  $\varepsilon$ -approximation of size  $O(\frac{1}{\varepsilon} \log^{2.5} \frac{1}{\varepsilon})$  exists for any  $P$  with respect to orthogonal ranges, and the best lower bound is  $\Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ .

The  $\varepsilon$ -approximation is a rather restricted data structure, as we are not allowed to store any information other than the coordinates of a subset of points in  $P$ . In this paper, we explore what can be achieved without any restriction on the data structure. We first describe a data structure that uses  $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \log n)$  bits that answers queries with error  $\varepsilon n$ . We then prove a lower bound that any data structure that answers queries with error  $O(\log n)$  must use  $\Omega(n \log n)$  bits. This lower bound has two consequences: 1) answering queries with error  $O(\log n)$  is as hard as answering the queries exactly; and 2) our upper bound cannot be improved in general by more than an  $O(\log \log \frac{1}{\varepsilon})$  factor.

## 1 Introduction

Range counting is one of the most fundamental problems in computational geometry and data structures. Given  $n$  points in  $d$  dimensions, the goal is to preprocess the points into a data structure, such that the number of points in a query range can be returned. Range counting has been studied intensively, and a lot of work has focused on the space-query time tradeoff or the update-query tradeoff of data structures. We refer the reader to the survey by Agarwal and Erickson [1] for these results. In this paper, we look at the problem from a data summarization/compression point of view: What is the minimum amount of space that is needed to encode all the range counts approximately? Approximation is necessary here, since otherwise we will have to remember the entire the point set. It is also easy to see that relative approximation will not help either, as it requires us to differentiate between empty ranges and those containing only one point. Thus, we aim at an absolute error guarantee. As we will be dealing with bit-level space complexity, it is convenient to focus on an integer grid. More formally, we are given a set of  $n$  points  $P$  drawn from an  $n \times n$  grid and an error parameter  $\varepsilon$ . The goal is to build a data structure, such that for any

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orthogonal range  $R$ , the data structure can return the number of points in  $P \cap R$  with additive error  $\varepsilon n$ .

We should mention that there is another notion of approximate range counting that approximates the range, i.e., points near the boundary of the range may or may not be counted [3]. Such an approximation notion clearly precludes any sublinear-space data structure as well.

## 1.1 Previous work

**$\varepsilon$ -approximations.** Summarizing point sets while preserving range counts (approximately) is a fundamental problem with applications in numerical integration, statistics, and data mining, among many others. The classical solution is to use the  $\varepsilon$ -*approximation* from discrepancy theory. Consider a range space  $(P, \mathcal{R})$ , where  $P$  is a finite point set of size  $n$ . A subset  $A \subseteq P$  is called an  $\varepsilon$ -approximation of  $(P, \mathcal{R})$  if

$$\max_{R \in \mathcal{R}} \left| \frac{|R \cap A|}{|A|} - \frac{|R \cap P|}{|P|} \right| \leq \varepsilon.$$

This means that we can approximate  $|R \cap P|$  by counting the number of points in  $R \cap A$  and scaling back, with error at most  $\varepsilon n$ .

Finding  $\varepsilon$ -approximations of small size for various geometric range spaces has been a central research topic in computational geometry. Please see the books by Matousek [14] and Chazelle [8] for comprehensive coverages on this topic. Here we only review the most relevant results, i.e., when the range space  $\mathcal{R}$  is all orthogonal rectangles in 2 dimensions. This question dates back to Beck [5], who showed that there are  $\varepsilon$ -approximations of size  $O(\frac{1}{\varepsilon} \log^4 \frac{1}{\varepsilon})$  for any point set  $P$ . This was later improved to  $O(\frac{1}{\varepsilon} \log^{2.5} \frac{1}{\varepsilon})$  by Srinivasan [20]. These were not constructive due to the use of a non-constructive coloring with combinatorial discrepancy  $O(\log^{2.5} n)$  for orthogonal rectangles. Recently, Bansal [4] proposed an algorithm to construct such a coloring, and therefore has made these results constructive. On the lower bound side, it is known that there are point sets that require  $\varepsilon$ -approximations of size  $\Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$  [5].

**Combinatorial discrepancy.** For a range space  $(P, \mathcal{R})$  and a coloring function  $\chi : P \rightarrow \{-1, +1\}$ , we write  $\chi(P \cap R) = \sum_{p \in P \cap R} \chi(p)$ . The *combinatorial discrepancy* of the range space  $(P, \mathcal{R})$  is defined as

$$\text{disc}(P, \mathcal{R}) = \min_{\chi} \max_{R \in \mathcal{R}} |\chi(P \cap R)|,$$

namely, we are looking at the coloring that minimizes the color difference of any range in  $\mathcal{R}$ . Taking the maximum over all point sets of size  $n$ , we say that the combinatorial discrepancy of  $\mathcal{R}$  is  $\text{disc}(n, \mathcal{R}) = \max_{|P|=n} \text{disc}(P, \mathcal{R})$ .

There is a close relationship between combinatorial discrepancy and  $\varepsilon$ -approximations, as observed by Beck [5]. For orthogonal ranges, the relationship is particularly simple: The combinatorial discrepancy is at most  $t(n)$  if and only if there is an  $\varepsilon$ -approximation of size  $O(\frac{1}{\varepsilon} t(\frac{1}{\varepsilon}))$ . In fact, all the aforementioned results on  $\varepsilon$ -approximations follow from the corresponding results on combinatorial discrepancy. So the current upper bound on the combinatorial discrepancy of orthogonal rectangles in 2 dimensions is  $O(\log^{2.5} n)$  [20]. The lower bound is  $\Omega(\log n)$  [5], which follows from the Lebesgue discrepancy lower bound (see below). Closing the  $\Theta(\log^{1.5} n)$  gap between the upper and the lower bound remains a major open problem in discrepancy theory. In  $d \geq 3$  dimensions, the current best upper bound is  $O(\log^{d+1/2} n)$  by Larsen [13], while the lower bound is  $\Omega((\log n)^{(d-1)/2+\eta})$ , where  $\eta = \eta(d)$  is a small constant depending on  $d$  [7].

**Lebesgue discrepancy.** Suppose the points of  $P$  are in the unit square  $[0, 1]^2$ . The Lebesgue discrepancy of  $(P, \mathcal{R})$  is defined to be

$$D(P, \mathcal{R}) = \sup_{R \in \mathcal{R}} \left| |P \cap R| - |R \cap [0, 1]^2| \right|.$$

The Lebesgue discrepancy describes how uniformly the point set  $P$  is distributed in  $[0, 1]^2$ . Taking the infimum over all point sets of size  $n$ , we say that the Lebesgue discrepancy of  $\mathcal{R}$  is  $D(n, \mathcal{R}) = \inf_{|P|=n} D(P, \mathcal{R})$ .

The Lebesgue discrepancy for orthogonal rectangles is known to be  $\Theta(\log n)$  in 2 dimensions. The lower bound is due to Schmidt [18], while there are many point sets (e.g., the Van der Corput sets [21] and the  $b$ -ary nets [19]) that are proved to have  $O(\log n)$  Lebesgue discrepancy. It is well known that the combinatorial discrepancy of a range space cannot be lower than its Lebesgue discrepancy, so this also gives the  $\Omega(\log n)$  lower bound on the combinatorial discrepancy of orthogonal rectangles mentioned above.

**Approximate range counting data structures.** The  $\varepsilon$ -approximation is a rather restricted data structure, as we are not allowed to store any information other than the coordinates of a subset of points in  $P$ . In this paper, we explore what can be achieved without any restriction on the data structure. In 1 dimension, there is nothing better: An  $\varepsilon$ -approximation has size  $O(\frac{1}{\varepsilon})$ , which takes  $O(\frac{1}{\varepsilon} \log n)$  bits. On the other hand, simply consider the case where the  $n$  points are divided into groups of size  $\varepsilon n$ , where all points in each group have the same location. There are  $n^{1/\varepsilon}$  such point sets and the data structure has to differentiate all of them. Thus  $\log(n^{1/\varepsilon}) = \frac{1}{\varepsilon} \log n$  is a lower bound on the number of bits used by the data structure.

## 1.2 Our results

This paper gives almost matching upper and lower bounds on the space needed by any data structure that encodes all the orthogonal range counts with error  $\varepsilon n$ . We first describe a data structure that uses  $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \log n)$  bits. This is a  $\Theta(\log^{1.5} \frac{1}{\varepsilon})$  (ignoring  $O(\log \log \frac{1}{\varepsilon})$  factors) improvement from  $\varepsilon$ -approximations. Of course, we will store some extra information other than the coordinates of the points.

On the lower bound side, we prove that there exists a constant  $c$ , such that any data structure that answers queries with error  $c \log n$  must use  $\Omega(n \log n)$  bits. This lower bound has two consequences: 1) answering queries with error  $O(\log n)$  is as hard as answering the queries exactly; and 2) our upper bound cannot be improved in general by more than an  $O(\log \log \frac{1}{\varepsilon})$  factor.

Our lower bound is information-theoretic: We show that there is a collection  $\mathcal{P}^*$  of  $2^{\Omega(n \log n)}$  point sets that are pairwise “different enough” in terms of range counts. More precisely, we show that the union of any two point sets in  $\mathcal{P}^*$  has high combinatorial discrepancy, i.e., at least  $c \log n$ . Then, for any two point sets  $P_1, P_2 \in \mathcal{P}^*$ , if  $\text{disc}(P_1 \cup P_2, \mathcal{R}_2) \geq c \log n$ , where  $\mathcal{R}_2$  is the set of all orthogonal rectangles, that means for any coloring  $\chi$  on  $P_1 \cup P_2$ , there must exist a rectangle  $R$  such that  $|\chi(R)| \geq c \log n$ . Consider the coloring  $\chi$  where  $\chi(p) = 1$  if  $p \in P_1$  and  $\chi(p) = -1$  if  $p \in P_2$ . Then there exists a rectangle  $R$  such that  $|\chi(R)| = ||R \cap P_1| - |R \cap P_2|| \geq c \log n$ . This implies that a data structure that answers queries with error  $\frac{c}{2} \log n$  have to distinguish  $P_1$  and  $P_2$ . To distinguish all the  $2^{\Omega(n \log n)}$  point sets in  $\mathcal{P}^*$ , the data structure has to use at least  $\Omega(n \log n)$  bits.

While point sets with low Lebesgue discrepancy or high combinatorial discrepancy have been extensively studied, we have constructed a large collection of point sets in which the pairwise

union has high combinatorial discrepancy. This particular aspect appears to be novel, and our construction could be useful in proving other data structure lower bounds. It may also have applications in situations where we need a “diverse” collection of random point sets.

## 2 Data Structure

In this section, we build a data structure that supports approximate range counting queries. Given a set of  $n$  points on an  $n \times n$  grid, our data structure uses  $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \log n)$  bits and answers an orthogonal range counting query with error  $\varepsilon n$ .

First note that it is sufficient to only consider two-sided ranges. A two-sided range is a rectangle of the form  $[0, x) \times [0, y)$ , where  $(x, y)$  is called the *query point*. By the inclusion-exclusion principle, a 4-sided range counting query can be expressed as a linear combination of four two-sided range counting queries, so a data structure that answers two-sided queries with error  $\varepsilon n/4$  can be used to answer 4-sided queries with error  $\varepsilon n$ .

Our data structure is based on an  $\varepsilon$ -net for orthogonal rectangles. For a range space  $(P, \mathcal{R})$ , a subset  $A \subseteq P$  is called an  $\varepsilon$ -net of  $P$  if for any range  $R \in \mathcal{R}$  that satisfies  $|P \cap R| \geq \varepsilon n$ , there is at least 1 point in  $A \cap R$ . Note that an  $\varepsilon$ -approximation is an  $\varepsilon$ -net, but the converse may not be true.

The size of the smallest  $\varepsilon$ -net for orthogonal rectangles in 2 dimensions has been recently settled. Aronv, Ezra and Sharir [2] gave an algorithm for constructing an  $\varepsilon$ -net of size  $O(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$  for any  $P$ , while a matching lower bound was given by Pach and Tardos [15].

**The data structure.** Consider a set  $P$  of  $n$  points on an  $n \times n$  grid. We first construct an  $\varepsilon$ -net  $P_\varepsilon$  of  $P$  for orthogonal rectangles. Let  $m = c \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}$  denote the size of this  $\varepsilon$ -net, for some constant  $c$ . For simplicity we assume  $m$  is a power of 2. Let  $\{u_i = (x_i, y_i) \mid i = 1, \dots, m\}$  denote the points in  $P_\varepsilon$ , in which the points are sorted on their  $x$ -coordinates. We call a point  $p \in P$  an  $\varepsilon$ -net point if  $p \in P_\varepsilon$ .

The key ingredient of our structure is a bit array attached to each  $\varepsilon$ -net point, which can be used to determine whether or not to include this point in our estimation during the query process. More precisely, After constructing an  $\varepsilon$ -net  $P_\varepsilon$ , we will associate a bit array  $C_i$  of  $\log m$  bits to each point  $u_i \in P_\varepsilon$ . The  $l$ -th bit of  $C_i$ , denoted  $C_i(l)$ , is called the  $l$ -indicator of  $u_i$ . To construct  $C_i$ , we need the following definition of *dyadic rectangles*:

**Definition 2.1.** For  $1 \leq l \leq \log m$ , we define dyadic rectangles at level  $l$  to be rectangles of the form  $[x_{a2^l}, x_{(a+1)2^l}) \times [0, y)$ , for  $a = 0, \dots, n/2^l - 1$  and  $y \geq 0$ . We use  $R_{a,l}^y$  to denote the dyadic rectangle  $[x_{a2^l}, x_{(a+1)2^l}) \times [0, y)$ . In particular, we use  $R_{a,l}^\infty$  denote the dyadic rectangle  $[x_{a2^l}, x_{(a+1)2^l}) \times [0, \infty)$ .

For  $1 \leq l \leq \log m$ , the  $l$ -indicators of the  $\varepsilon$ -net points are determined as follow. Let  $R_{a,l}^\infty = [x_{a2^l}, x_{(a+1)2^l}) \times [0, \infty)$  be a dyadic rectangle at level  $l$ . We use  $P_{a,l} = \{p_1, \dots, p_s\}$  to denote the set of points of  $P$  in  $R_{a,l}^\infty$ , in which the points are sorted in their  $y$ -coordinates. We divide  $P_{a,l}$  into chunks of size  $\varepsilon n$ . For each chunk, we find the lowest  $\varepsilon$ -net point (there must exist one due to the  $\varepsilon$ -net property, except possibly the highest chunk if it contains less than  $\varepsilon n$  points in  $P$ ) and set its  $l$ -indicator to 1. The  $l$ -indicators of the other  $\varepsilon$ -net points in  $P_{a,l}$  are set to 0. See Figure 1. Then we do so for each dyadic rectangle  $R_{a,l}^\infty$  and for all  $1 \leq l \leq \log m$ .

**Query process.** Given a query point  $q = (q_x, q_y)$ , we initialize a counter  $Q = 0$ . Let  $u_{i_q}$  be the  $\varepsilon$ -net point such that  $x_{i_q} \leq q_x < x_{i_q+1}$ . We note that the range  $[0, x_{i_q}) \times [0, q_y)$  can be decomposed

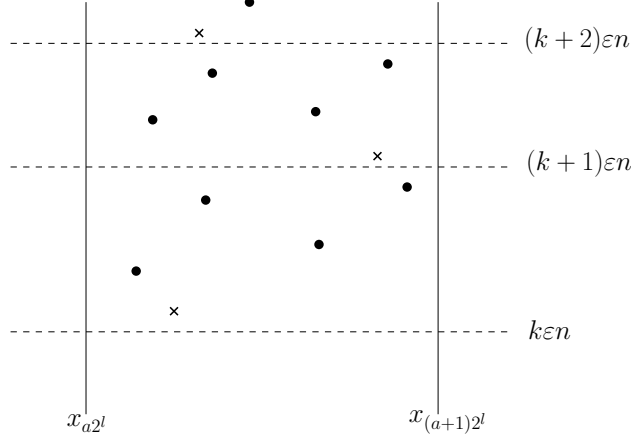


Figure 1: Illustration of the construction of the  $l$ -indicators. The solid dots represent  $\epsilon$ -net points, and the crosses represent  $\epsilon$ -net points with  $l$ -indicators set to 1.

into at most  $\log m$  dyadic rectangles, and we use  $\mathcal{R}_q$  denote the set of these dyadic rectangles. For each dyadic rectangle  $R_{a,l}^{q_y} \in \mathcal{R}_q$  and  $\epsilon$ -net point  $u_i \in R_{a,l}^{q_y}$ , we add  $\epsilon n$  to the counter  $Q$  if  $C_i(l) = 1$ . After all dyadic rectangles in  $\mathcal{R}$  have been processed, we have

$$Q = \sum_{R_{a,l}^{q_y} \in \mathcal{R}_q} \sum_{u_i \in P_\epsilon \cap R_{a,l}^{q_y}} \epsilon n \cdot C_i(l).$$

Then the data structure returns  $Q$  as the approximated count. Since the focus of the paper is on the space complexity, we have ignored query time. In fact, by precomputing some appropriate prefix counts on the  $C_i(l)$ 's,  $Q$  can be computed in  $O(\log \frac{1}{\epsilon} + \log \log n)$  time; we omit the details to avoid digression.

**Analysis.** We now analyze the error of the above estimation. We claim that  $Q$  estimates the number of points of  $P$  inside  $[0, q_x) \times [0, q_y)$  with error at most  $(\log m + 1)\epsilon n$ . The following lemma gives an upper bound of the error in a single dyadic rectangle.

**Lemma 2.1.** *Given a dyadic rectangle  $R_{a,l}^y$ , The following inequality holds:*

$$0 \leq \sum_{u_i \in P_\epsilon \cap R_{a,l}^y} \epsilon n \cdot C_i(l) - |P \cap R_{a,l}^y| \leq \epsilon n.$$

*Proof.* Recall that we divided the points in  $P \cap R_{a,l}^\infty$  into chunks of size  $\epsilon n$ , and set the  $l$ -indicator of the lowest  $\epsilon$ -net point to be 1. Thus, each such  $\epsilon$ -net point represents exactly  $\epsilon n$  points in  $P \cap R_{a,l}^y$ , except the highest one, which may represent less than  $\epsilon n$  points. The lemma then follows.  $\square$

Since there are at most  $\log m$  dyadic rectangles in  $\mathcal{R}_q$ ,  $Q$  is an estimator of  $|P \cap [0, u_{i_q}] \times [0, q_y)|$  with error at most  $\epsilon n \log m$ . We also note that there is no  $\epsilon$ -net point in  $(u_{i_q}, q_x) \times [0, q_y)$ , so the error from this rectangle is at most  $\epsilon n$ . This proves that  $Q$  estimates  $|P \cap [0, q_x) \times [0, q_y)|$  with error  $(\log m + 1)\epsilon n = (\log(c \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}) + 1)\epsilon n$ . It is easy to see that our data structure uses  $O(\frac{1}{\epsilon} \log \log \frac{1}{\epsilon} \log n)$  bits, which is dominated by the coordinates of the  $\epsilon$ -net points. By using  $\epsilon' = \epsilon / (\log(c \frac{1}{\epsilon} \log \log \frac{1}{\epsilon}) + 1)$  in place of  $\epsilon$  in the construction, we obtain a data structure that uses  $O(\frac{1}{\epsilon'} \log \frac{1}{\epsilon'} \log \log \frac{1}{\epsilon'} \log n)$  bits and answers two-sided range counting queries with error  $\epsilon' n$ .

**Theorem 2.1.** *Given a set of  $n$  points drawn from an  $n \times n$  grid, there is a data structure that uses  $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \log n)$  bits and answers orthogonal range counting query with additive error  $\varepsilon n$ .*

### 3 Lower Bound

As argued in the introduction, our lower bound follows from the following theorem.

**Theorem 3.1.** *Let  $\mathcal{P}$  denote the collection of all  $n$ -point sets drawn from an  $n \times n$  grid. There exists a constant  $c$  and a sub-collection  $\mathcal{P}^* \subseteq \mathcal{P}$  of size  $2^{\Omega(n \log n)}$ , such that for any  $P_1, P_2 \in \mathcal{P}^*$ ,  $\text{disc}(P_1 \cup P_2, \mathcal{R}_2) \geq c \log n$ .*

In the rest of this section, we will focus on proving Theorem 3.1. We will use the a collection of point sets called *binary nets* as a ground set to derive the point sets that satisfy the conditions in Theorem 3.1. Binary nets are a special type of point sets under a more general concept called  $(t, m, s)$ -nets, which are introduced in [14] as an example of point sets with low Lebesgue discrepancy. See the survey by Clayman et. al. [9] or the book by Hellekalek et. al. [12] for more results on  $(t, m, s)$ -nets. In this paper we will show that binary nets have two other nice properties: 1) A binary net has high combinatorial discrepancy, i.e.,  $\Omega(\log n)$ ; 2) there is a bit vector representation of every binary net, which allows us to extract a sub-collection by constructing a subset of bit vectors. In the following sections, we will define binary nets, and formalize these two properties.

#### 3.1 Definitions

For ease of the presentation, we assume that the  $n \times n$  grid is embedded in the square  $[0, n]^2$ . We partition  $[0, n]^2$  into  $n \times n$  unit squares. We assume the grid points are placed at the mass centers of the  $n^2$  unit squares, that is, each grid point has coordinates  $(i + 1/2, j + 1/2)$ , for  $i, j \in [n]$ , where  $[n]$  denote the set of all integers in  $[0, n)$ . For the sake of simplicity, we define the grid point  $(i, j)$  to be the grid point with coordinates  $(i + 1/2, j + 1/2)$ , and we do not distinguish a grid point and the square it resides in.

Now we introduce the concepts of  $(a, b)$ -cell and  $k$ -canonical cell.

**Definition 3.1.** *A  $(a, b)$ -cell at position  $(i, j)$  is the rectangle  $[i2^a, (i + 1)2^a) \times [j2^b, (j + 1)2^b)$ . We use  $G_{a,b}(i, j)$  to denote the  $(a, b)$ -cell at position  $(i, j)$ , and  $G_{a,b}$  to denote the set of all  $(a, b)$ -cells.*

**Definition 3.2.** *A  $k$ -canonical cell at position  $(i, j)$  is a  $(k, \log n - k)$ -cell with coordinates  $(i, j)$ . We use  $G_k(i, j)$ , to denote the  $k$ -canonical cell at position  $(i, j)$ , and  $G_k$  to denote the set of all  $k$ -canonical cells.*

Figure 2 is the illustration of  $(a, b)$ -cells and canonical cells. Note that the position  $(i, j)$  for a  $(a, b)$ -cell takes value in  $[n/2^a] \times [n/2^b]$ . In particular, we call  $G_0(i, 0)$  the  $i$ -th column and  $G_{\log n}(0, j)$  the  $j$ -th row. Note that for a fixed  $k$ ,  $G_k$  partitions the grid  $[0, n]^2$  into  $n$  rectangles. Based on the definition of  $k$ -canonical cells, we define the binary nets:

**Definition 3.3.** *A point set  $P$  is called a binary net if for any  $k \in [\log n]$ ,  $P$  has exactly one point in each  $k$ -canonical cell.*

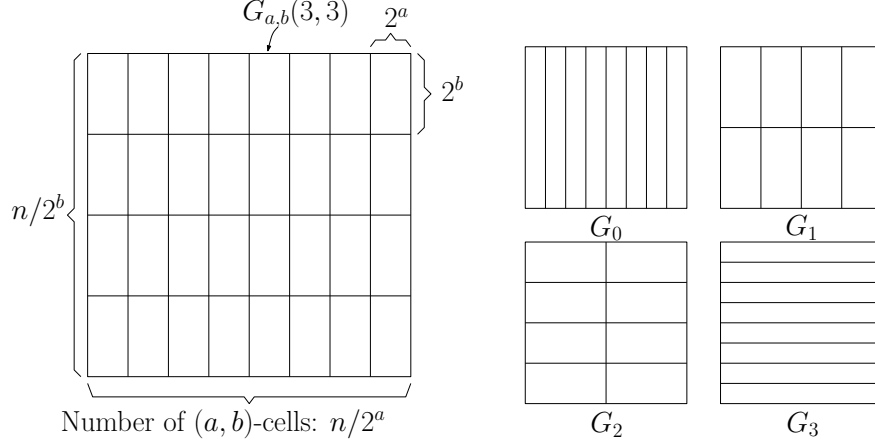


Figure 2: Illustrations of  $(a, b)$ -cells and canonical cells.

Let  $\mathcal{P}_0$  denote the collection of binary nets. In other word,  $\mathcal{P}_0$  is the set

$$\{P \mid |P \cap G_k(i, j)| = 1, k \in [\log n], i \in [n/2^k], j \in [2^k]\}.$$

It is known that the point sets in  $\mathcal{P}_0$  have Lebesgue discrepancy  $O(\log n)$ ; below we show that they also have  $\Omega(\log n)$  combinatorial discrepancy. However, the union of two point sets in  $\mathcal{P}_0$  could have combinatorial discrepancy as low as  $O(1)$ . Thus we need to carefully extract a subset from  $\mathcal{P}_0$  with high pairwise union discrepancy.

### 3.2 Combinatorial Discrepancy and Corner Volume

We first prove that the combinatorial discrepancy of any point set in  $\mathcal{P}_0$  is large. To show this, we need the following definition of *corner volume*:

**Definition 3.4.** For a point set  $P \in \mathcal{P}_0$  and a  $k$ -canonical cell  $G_k(i, j)$ , let  $q$  be the point in  $P \cap G_k(i, j)$ . Define the corner volume  $V_P(k, i, j)$  to be the volume of the orthogonal rectangle defined by  $q$  and its nearest corner of  $G_k(i, j)$ . We use  $S_P$  to denote the summation of the corner volumes over all possible triples  $(k, i, j)$ , that is,

$$S_P = \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} V_P(k, i, j).$$

See Figure 3 for the illustration of corner volumes. The following lemma relates the combinatorial discrepancy of  $P$  with its corner volume sum  $S_P$ .

**Lemma 3.1.** *There exists a constant  $c$ , such that for any point set  $P \in \mathcal{P}_0$  that satisfies*

$$S_P \geq cn^2 \log n,$$

*we have  $\text{disc}(P, \mathcal{R}_2) = \Omega(\log n)$ .*

By Lemma 3.1, we can prove that the combinatorial discrepancy of a binary net is large by showing its corner volume sum is large. In particular, we have the following lemma:

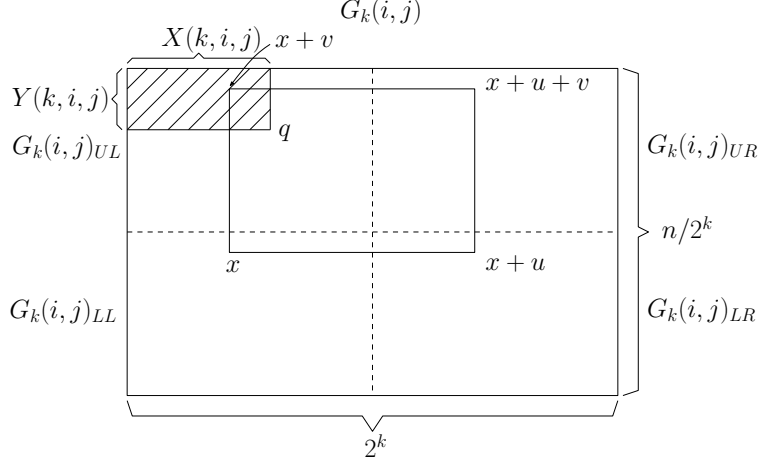


Figure 3: Illustration of the corner volume and the four analogous points. The area in shadow represents the corner volume  $V_P(k, i, j)$ .

**Lemma 3.2.** *For any point set  $P \in \mathcal{P}_0$ , we have  $\text{disc}(P, \mathcal{R}_2) = \Omega(\log n)$ .*

Strictly speaking, Theorem 3.1 does not depend on Lemma 3.2 and Lemma 3.1, but these lemmas give us some insight on the binary nets. Moreover, a key lemma to proving Theorem 3.1 (Lemma 3.4) shares essentially the same proof with Lemma 3.1. We defer the proof of Lemma 3.1 and Lemma 3.2 to the Appendix in order to avoid digression into technicalities.

### 3.3 A bit vector representation for $\mathcal{P}_0$

Another nice property of  $\mathcal{P}_0$  is that we can derive the exact number of point sets in it. The following lemma is from the book [10]:

**Lemma 3.3** ([10]). *The number of point sets in  $\mathcal{P}_0$  is  $2^{\frac{1}{2}n \log n}$ .*

We sketch the proof of Lemma 3.3 here, as it provides a bit vector presentation of each binary net, which is essential in our lower bound proof.

*Proof of Lemma 3.3.* It is equivalent to prove that the number of possible ways to place  $n$  points on the  $n \times n$  grid such that any  $k$ -canonical cell  $G_k(i, j)$  has exactly 1 point is  $2^{\frac{1}{2}n \log n}$ . We prove it by induction on  $n$ . Let  $\mathcal{P}_0(n)$  denote the collection of binary nets of size  $n$  in a  $n \times n$  grid.

Observe that the line  $y = n/2$  divides the grid  $[0, n]^2$  into two rectangles: the upper grid  $[0, n] \times [n/2, n)$  and the lower grid  $[0, n] \times [0, n/2)$ . For  $i$  even, let  $R_i$  denote the rectangle defined by the union of  $i$ -th and  $(i + 1)$ -th columns  $G_0(i, 0)$  and  $G_0(i + 1, 0)$ . Note that the line  $y = n/2$  divides  $R_i$  into  $G_1(i/2, 0)$  and  $G_1(i/2, 1)$ , and therefore defines four quadrants. By the definition of  $\mathcal{P}_0$ , for any point set  $P \in \mathcal{P}_0$ , the two points in  $G_0(i, 0)$  and  $G_1(i + 1, 0)$  must either reside in the lower left and upper right quadrants or in the lower right and upper left quadrants. There are in total  $n/2$  even  $i$ 's, so the number of the possible choices is  $2^{n/2}$ . See Figure 4. Note that after determining which half the point in each column resides in, the problem is divided into two sub-problems: counting the number of possible ways to place  $n/2$  points in the upper grid and the



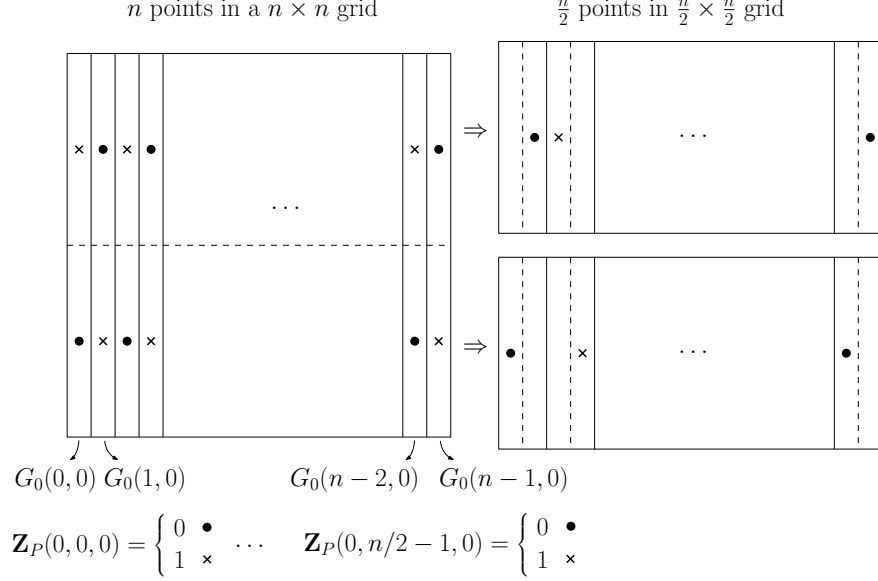


Figure 4: Illustration of the partition vector of  $G_0$ .

lower grid. It is easy to show that each sub-problem is identical to the problem of counting the number of point sets in  $\mathcal{P}_0(n/2)$ , so we have the following recursion:

$$|\mathcal{P}_0(n)| = 2^{\frac{n}{2}} \cdot |\mathcal{P}_0(n/2)|^2.$$

Solving this recursion with  $\mathcal{P}_0(1) = 1$  yields that  $|\mathcal{P}_0(n)| = 2^{\frac{1}{2}n \log n}$ .  $\square$

A critical observation is that the proof of Lemma 3.3 actually reveals a bit vector representation for each of the point sets in  $\mathcal{P}_0$ , which will allow us to refine the collection  $\mathcal{P}_0$ . To see this, we define the *partition vector*  $\mathbf{Z}_P$  for a point set  $P \in \mathcal{P}_0$  as follows. For any  $(k, i, j) \in [\log n] \times [n/2^{k+1}] \times [2^k]$ , consider the  $k$ -canonical cells  $G_k(2i, j)$  and  $G_k(2i+1, j)$  and  $(k+1)$ -canonical cells  $G_{k+1}(i, 2j)$  and  $G_{k+1}(i, 2j+1)$ . The two  $k$ -canonical cells overlap with the two  $(k+1)$ -canonical cells, which defines four quadrants. By the definition of binary nets, there are two points in  $P$  contained in these quadrants. We define  $\mathbf{Z}_P(k, i, j) = 0$  if the two points are in the lower left and upper right quadrants and  $\mathbf{Z}_P(k, i, j) = 1$  if they are in the lower right and upper left quadrants. See Figure 4. We say the  $k$ -canonical cells  $G_k(2i, j)$  and  $G_k(2i+1, j)$  is associated with bit  $\mathbf{Z}_P(k, i, j)$ . Note that we use the triple  $(k, i, j)$  as the index into  $\mathbf{Z}_P$  for the ease of presentation; we can assume that the bits in  $\mathbf{Z}_P$  are stored in for example the lexicographic order of  $(k, i, j)$ . Since the number of triples  $(k, i, j)$  is  $\frac{1}{2}n \log n$ , the total number of bits in  $\mathbf{Z}_P$  is  $\frac{1}{2}n \log n$ . Let  $\mathcal{Z}_0 = \{0, 1\}^{\frac{1}{2}n \log n}$  denote the set of all possible partition vector  $\mathbf{Z}_P$ 's. By the proof of Lemma 3.3, there is a bijection between  $\mathcal{Z}_0$  and  $\mathcal{P}_0$ .

### 3.4 Combinatorial discrepancy and corner volume distance

Although we have proved that binary nets have large combinatorial discrepancy, it does not yet lead us to Theorem 3.1. In this section, we will refine  $\mathcal{P}_0$ , the collection of all binary nets, to derive a collection  $\mathcal{P}^*$ , such that the union of any two point sets in  $\mathcal{P}^*$  has large combinatorial discrepancy.

In order to characterize the combinatorial discrepancy of the union of two point sets, we will need the following definition of *corner volume distance*.

**Definition 3.5.** For two point sets  $P_1, P_2 \in \mathcal{P}_0$ , the corner volume distance of  $P_1$  and  $P_2$  is the summation of  $|V_{P_1}(k, i, j) - V_{P_2}(k, i, j)|$ , over all  $(k, i, j)$ . In other words, let  $\Delta(P_1, P_2)$  denote the corner volume distance of  $P_1$  and  $P_2$ , then

$$\Delta(P_1, P_2) = \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k, i, j) - V_{P_2}(k, i, j)|.$$

The following lemma (proof in the Appendix) relates the combinatorial discrepancy of the union of two point sets with their corner volume distance:

**Lemma 3.4.** Let  $\mathcal{P}^*$  be a subset of  $\mathcal{P}_0$ . If there exists a constant  $c$ , such that for any two point sets  $P_1, P_2 \in \mathcal{P}_0$ , that their corner volume distance satisfies  $\Delta(P_1, P_2) \geq cn^2 \log n$ , then  $\text{disc}(P_1 \cup P_2, \mathcal{R}_2) = \Omega(\log n)$ .

Here we briefly explain the high level idea for proving Theorem 3.1. By Lemma 3.4, it is sufficient to find a sub-collection  $\mathcal{P}^* \subseteq \mathcal{P}_0$ , such that for any two point sets in  $\mathcal{P}^*$ , their corner volume distance is large. We will choose a subset  $\mathcal{Z}_1 \subseteq \mathcal{Z}_0$ , and project each vector in  $\mathcal{Z}_1$  down to a slightly shorter bit vector  $\mathbf{T}$ . The collection  $\mathcal{T}$  of all resulted bit vector  $\mathbf{T}$ 's induces a sub-collection  $\mathcal{P}_1 \subseteq \mathcal{P}_0$ , and each  $\mathbf{T}$  represents a point set in  $\mathcal{P}_1$ . Then we prove that for any two point sets  $P_1, P_2 \in \mathcal{P}_1$ , there is a linear dependence between the corner volume distance  $\Delta(P_1, P_2)$  and the hamming distance of their bit vector representations  $\mathbf{T}_{P_1}$  and  $\mathbf{T}_{P_2}$ . Finally, we show that there is a large sub-collection of  $\mathcal{T}$  with large pair-wise hamming distances, and this sub-collection induces a collection of point sets  $\mathcal{P}^* \in \mathcal{P}_1$  in which the union of any two point sets has large combinatorial discrepancy.

We focus on a  $(k+6, \log n - k)$ -cell  $G_{k+6, \log n - k}(i, j)$ , for  $k \in \{0, 6, 12, \dots, \log n - 6\}$ . Note that  $G_{k+6, \log n - k}(i, j)$  only contains  $(k+l)$ -canonical cells for  $l \in [7]$ . Let  $F_{k, i, j}(l)$  denote the set of all  $(k+l)$ -canonical cells in  $G_{k+6, \log n - k}(i, j)$ , it is easy to see that

$$F_{k, i, j}(l) = \{G_{k+l}(2^{6-l}i + s, 2^l j + t) \mid s \in [2^{6-l}], t \in [2^l]\}.$$

Note that  $|F_{k, i, j}(l)| = 64$  for each  $l \in [7]$ . Let  $Z_{k, i, j}(l)$  denote the set of indices of bits in the partition vector that are associated with the some  $(k+l)$ -canonical cells in  $G_{k+6, \log n - k}(i, j)$ , for  $l \in [6]$ , i.e.,

$$Z_{k, i, j}(l) = \{(k+l, 2^{5-l}i + s, 2^l j + t) \mid s \in [2^{5-l}], t \in [2^l]\}.$$

Define  $Z_{k, i, j}$  to be the union of the  $Z_{k, i, j}(l)$ 's. Since there are 32 bits in  $Z_{k, i, j}(l)$  for each  $l \in [6]$ , the total number of bits in  $Z_{k, i, j}$  is 192 (here we use the indices in  $Z_{k, i, j}$  to denote their corresponding bits in the partition vector of  $P$ , with a slightly abuse of notation). The following fact shows the  $Z_{k, i, j}$ 's partition all the  $\frac{1}{2}n \log n$  bits:

**Fact 3.1.** The number of  $Z_{k, i, j}$ 's is  $\frac{1}{384}n \log n$ ; For different  $(k, i, j)$  and  $(k', i', j')$ ,  $Z_{k, i, j} \cap Z_{k', i', j'} = \emptyset$ .

The proof of the above claims are fairly straightforward: The number of different  $Z_{k, i, j}$ 's is equal to the number of different  $G_{k+6, \log n - k}(i, j)$ 's. For a fixed  $k$ , the number of different  $(k+6, \log n - k)$ -cells is  $n/64$ , and the number of different  $k$ 's is  $\log n/6$ , so the total number of different  $Z_{k, i, j}$ 's is

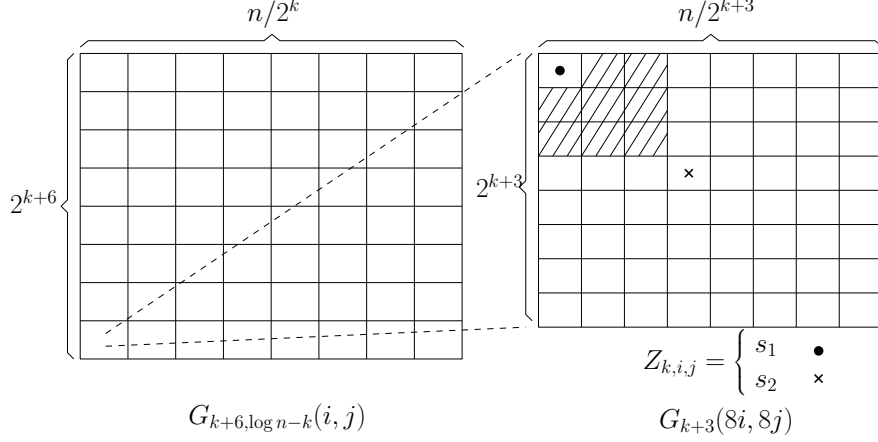


Figure 5: Illustration of the  $64 \times 64$  grid. The volume of each cell in  $G_{k+3}(8i, 8j)$  is  $n/64$ . The cells in shadow represent the corner volume difference of  $s_1$  and  $s_2$ .

$\frac{1}{384}n \log n$ . For the second claim, we consider the following two cases: If  $k = k'$ , we have  $(i, j) \neq (i', j')$ . This implies that the two  $(k, \log n - k + 6)$ -cells are disjoint, therefore the bits associated with the canonical cells inside them are disjoint. For  $k \neq k'$ , observe that we choose  $k$  and  $k'$  from  $\{0, 6, \dots, \log n - 6\}$ , and  $Z_{k,i,j}$  and  $Z_{k',i',j'}$  only contain bits associated with  $(k+l)$ -canonical cells and  $(k'+l')$ -canonical cells, respectively, for  $l, l' \in [6]$ , so  $Z_{k,i,j}(l)$  and  $Z_{k',i',j'}(l')$  are disjoint, for  $l, l' \in [6]$ .

The reason we group the bits in the partition vector into small subsets is that we can view each subset  $Z_{k,i,j}$  as a partition vector of the cell  $G_{k+6, \log n - k}(i, j)$ , which allows us to manipulate the positions of the points inside it. More precisely, we can view  $G_{k+6, \log n - k}(i, j)$  as a  $64 \times 64$  grid, with each grid cell being a  $(k, \log n - k - 6)$ -cell in the original  $[0, n]^2$  grid. Moreover, a  $(k+l)$ -canonical cell contained in  $G_{k+6, \log n - k}(i, j)$  corresponds to a  $l$ -canonical cell in the  $64 \times 64$  grid. Note that there are 64 points in this grid, and the bits in  $Z_{k,i,j}$  correspond to the partition vector of this 64-point set. Now consider a  $(k+3)$ -canonical cell  $G_{k+3}(8i, 8j)$ , which corresponds to the lower left  $8 \times 8$  grid in  $G_{k+6, \log n - k}(i, j)$ . For each point set  $P \in \mathcal{P}_0$ , there is exactly one point in  $G_{k+3}(8i, 8j)$ , and the bits in  $Z_{k,i,j}$  encode the position of the point on the  $8 \times 8$  grid. Suppose  $s_1$  and  $s_2$  are two bit vectors of length 192, such that when the bits in  $Z_{k,i,j}$  are assigned as  $s_1$  (denoted  $Z_{k,i,j} = s_1$ ), the point in  $G_{k+3}(8i, 8j)$  resides in the upper left grid cell,; and when  $Z_{k,i,j} = s_2$ , it resides in the grid cell to the upper left of the center of  $G_{k+3}(8i, 8j)$  (see Figure 5). Note that by this definition, the corner volume distance of this two point is at least  $n/8$ . Meanwhile, since there are no constraints on the other 63 points in  $G_{k+6, \log n - k}(i, j)$ , it is easy to show that such assignments  $s_1$  and  $s_2$  indeed exist.

By restricting the assignments of  $Z_{k,i,j}$  to  $\{s_1, s_2\}$ , we have created a subset  $\mathcal{Z}_1$  of  $\mathcal{Z}_0 = \{0, 1\}^{\frac{1}{2}n \log n}$ .

$$\mathcal{Z}_1 = \{\mathbf{Z} \mid Z_{k,i,j} = s_1 \text{ or } s_2, k \in \{0, 6, \dots, \log n - 6\}, i \in [n/2^{k+6}], j \in [2^k]\}.$$

Let  $\mathcal{P}_1$  denote the sub-collection of  $\mathcal{P}_0$  that  $\mathcal{Z}_1$  encode. By Fact 3.1, the number of  $Z_{k,i,j}$ 's is  $\frac{1}{384}n \log n$ , so  $|\mathcal{P}_1| = 2^{\frac{1}{384}n \log n}$ . Define a bit vector  $\mathbf{T}$  of length  $\frac{1}{384}n \log n$ , such that  $\mathbf{T}(k, i, j) = 0$  if  $Z_{k,i,j} = s_1$  and  $\mathbf{T}(k, i, j) = 1$  if  $Z_{k,i,j} = s_2$ , then a bit vector  $\mathbf{T}$  encodes a bit vector  $\mathbf{Z} \in \mathcal{Z}_1$ , and

therefore encodes a point set in  $\mathcal{P}_1$ . Let  $\mathcal{T} = \{0, 1\}^{\frac{1}{384}n \log n}$  denote the collection of all bit vectors  $\mathbf{T}$ . Then there is a bijection between  $\mathcal{T}$  and  $\mathcal{P}_1$ , and  $|\mathcal{T}| = |\mathcal{P}_1| = 2^{\frac{1}{384}n \log n}$ .

Consider two point sets  $P_1$  and  $P_2$  in  $\mathcal{P}_1$ . Let  $\mathbf{T}_{P_1}$  and  $\mathbf{T}_{P_2}$  denote the bit vector that encode these two point sets, respectively. The following lemma relates the corner volume distance of  $P_1$  and  $P_2$  with the hamming distance between  $\mathbf{T}_{P_1}$  and  $\mathbf{T}_{P_2}$ .

**Lemma 3.5.** *Suppose there exists a constant  $c$ , such that for any  $P_1, P_2 \in \mathcal{P}_1$ , the hamming distance  $H(\mathbf{T}_{P_1}, \mathbf{T}_{P_2}) \geq cn \log n$ , then the corner volume distance between  $P_1$  and  $P_2$ ,  $\Delta(P_1, P_2)$ , is  $\Omega(n^2 \log n)$ .*

*Proof.* We make the following relaxation on  $\Delta(P_1, P_2)$  :

$$\begin{aligned} \Delta(P_1, P_2) &= \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k, i, j) - V_{P_2}(k, i, j)| \\ &\geq \sum_{k \in \{0, 6, \dots, \log n - 6\}} \sum_{i=0}^{n/2^{k+6}-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k+3, 8i, 8j) - V_{P_2}(k+3, 8i, 8j)|. \end{aligned}$$

Now consider the bits  $\mathbf{T}_{P_1}(k, i, j)$  and  $\mathbf{T}_{P_2}(k, i, j)$ . If  $\mathbf{T}_{P_1}(k, i, j) \neq \mathbf{T}_{P_2}(k, i, j)$ , then by the choice of  $s_1$  and  $s_2$  we have  $|V_{P_1}(k+3, 8i, 8j) - V_{P_2}(k+3, 8i, 8j)| \geq n/8$ . So the corner volume distance  $\Delta(P_1, P_2)$  is lower bounded by the hamming distance  $H(\mathbf{T}_{P_1}, \mathbf{T}_{P_2})$  multiplied by  $n/8$ , and the lemma follows.  $\square$

The following lemma (probably folklore; see a proof in the Appendix) states that there is a large subset of  $\mathcal{T}$ , in which the vectors are well separated in terms of hamming distance.

**Lemma 3.6.** *Let  $N = \frac{1}{384}n \log n$ . There is a subset  $\mathcal{T}^* \subseteq \mathcal{T} = \{0, 1\}^N$  of size  $2^{\frac{1}{16}N}$ , such that for any  $\mathbf{T}_1 \neq \mathbf{T}_2 \in \mathcal{T}^*$ , the hamming distance  $H(\mathbf{T}_1, \mathbf{T}_2) \geq \frac{1}{4}N$ .*

*Proof.* We embed  $\mathcal{T}$  into a graph  $(V, E)$ . Each node in  $V$  represents a vector  $\mathbf{T} \in \mathcal{T}$ , and there is edge between two nodes  $\mathbf{T}_1$  and  $\mathbf{T}_2$  if and only if  $H(\mathbf{T}_1, \mathbf{T}_2) < \frac{1}{4}N$ . By this embedding, it is equivalent to prove that there is an independent set of size  $2^{\frac{1}{16}N}$  in  $(V, E)$ .

Fix a vector  $\mathbf{T} \in \mathcal{T}$ , and consider a random vector  $\mathbf{T}'$  uniformly drawn from  $\mathcal{T}$ . It is easy to see that the hamming distance  $H(\mathbf{T}, \mathbf{T}')$  follows binomial distribution. By Chernoff bound

$$\Pr[H(\mathbf{T}, \mathbf{T}') < \frac{1}{4}N] \leq e^{-\frac{1}{16}N} \leq 2^{-\frac{1}{16}N}.$$

This implies that the probability that there is an edge between  $\mathbf{T}$  and  $\mathbf{T}'$  is at most  $2^{-\frac{1}{16}N}$ . By the fact that  $\mathbf{T}'$  is uniformly chosen from  $\mathcal{T}$ , it follows that the degree of  $\mathbf{T}$  is at most  $d = 2^N \cdot 2^{-\frac{1}{16}N} = 2^{\frac{15}{16}N}$ . Since a graph with maximum degree  $d$  must have an independent set of size at least  $|V|/d$ , there must be an independent set of size at least  $2^{\frac{1}{16}N}$ .  $\square$

Let  $\mathcal{P}^*$  denote the collection of point sets encoded by  $\mathcal{T}^*$ . By Lemma 3.6,  $|\mathcal{P}^*| \geq 2^{\frac{1}{16}N} = 2^{\frac{1}{6144}n \log n}$ . From Lemma 3.4 and Lemma 3.5 we know that for any two point sets  $P_1, \neq P_2 \in \mathcal{P}^*$ , the combinatorial discrepancy of the union of  $P_1$  and  $P_2$  is  $\Omega(\log n)$ . This completes the proof of Theorem 3.1.

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# A Appendix

## A.1 Proof of Lemma 3.1

*Proof.* The proof makes use of the Roth's orthogonal function method [16], which is widely used for proving lower bounds for Lebesgue discrepancy (see [8, 14]). Consider any point set  $P \in \mathcal{P}_0$  that satisfies  $S_P \geq cn^2 \log n$ . Given any coloring  $\chi : P \rightarrow \{-1, +1\}$  and a point  $x = (x_1, x_2) \in [0, n]^2$ , the combinatorial discrepancy  $D(x)$  at a point  $x$  is defined to be

$$D(x) = \sum_{p \in P \cap [0, x_1] \times [0, x_2]} \chi(p).$$

If we can prove  $\sup_{x \in [0, n]^2} |D(x)| = \Omega(\log n)$ , the lemma will follow.

For  $k \in [\log n]$ , define normalized wavelet functions  $f_k$  as follow: for each  $k$ -canonical cell  $G_k(i, j)$ , let  $q$  denote the point contained in it. We subdivide  $G_k(i, j)$  into four equal-size quadrants, and use  $G_k(i, j)_{UR}$ ,  $G_k(i, j)_{UL}$ ,  $G_k(i, j)_{LR}$ ,  $G_k(i, j)_{LL}$  to denote the upper right, upper left, lower right and lower left quadrants, respectively (See Figure 3). Set  $f_k(x) = \chi(q)/n^2$  over quadrants  $G_k(i, j)_{UR}$  and  $G_k(i, j)_{LL}$ , and  $f_k(x) = -\chi(q)/n^2$  over the other two quadrants. We say a function  $f : [0, n]^2 \rightarrow \mathbb{R}$  is  $(a, b)$ -checkered if for each  $(a, b)$ -cell  $G_{a,b}(i, j)$ , there exists a color  $C \in \{-1, +1\}$  such that  $f$  is equal to  $C/n^2$  over  $G_{a,b}(i, j)_{UR}$  and  $G_{a,b}(i, j)_{LL}$ , and  $f(x) = -C/n^2$  over the other two quadrants. Note that the wavelet function  $f_k$  is  $(k, \log n - k)$ -checkered. It is easy to see that the integration of a  $(a, b)$ -checkered function over an  $(a, b)$ -cell is 0. The following lemma is proved in the book [8]:

**Lemma A.1** ([8]). *If  $f$  is  $(a, b)$ -checkered and  $g$  is  $(a', b')$  checkered, where  $a < a'$  and  $b' < b$ , then  $fg$  is  $(a, b')$ -checkered.*

The following corollary can be directly derived from Lemma A.1:

**Corollary A.1.** *For  $0 \leq k_1 < \dots < k_l \leq \log n$ , the function  $f_{k_1}(x) \cdots f_{k_l}(x)$  is a  $(k_1, \log n - k_l)$ -checkered. As a consequence, we have*

$$\int f_{k_1} \cdots f_{k_l} = 0.$$

Note that the range of the integration is  $[0, n]^2$  and the variable of integration is  $dx$  when not specified. We define the Riesz product

$$G(x) = -1 + \prod_{k=0}^{\log n} (\gamma f_k(x) + 1),$$

where  $\gamma$  is some constant to be determined later. By the inequality

$$\left| \int GD \right| \leq \int |GD| \leq \sup_{x \in [0, n]^2} |D| \cdot \int |G|,$$

we can lower bound the combinatorial discrepancy of  $P$  as follow:

$$\sup_{x \in [0, n]^2} |D| \geq \left| \int GD \right| / \int |G|. \tag{A.1}$$

For the denominator, we have

$$\begin{aligned}
\int |G| &= \int \left| -1 + \prod_{k=0}^{\log n} (\gamma f_k + 1) \right| \\
&\leq 1 + \sum_{l=0}^{\log n} \gamma^l \sum_{0 \leq k_1 < \dots < k_l \leq \log n} \int f_{k_1} \cdots f_{k_l} \\
&= 2 + \sum_{l=1}^{\log n} \gamma^l \sum_{0 \leq k_1 < \dots < k_l \leq \log n} \int f_{k_1} \cdots f_{k_l} \\
&= 2.
\end{aligned} \tag{A.2}$$

The last equation is due to Corollary A.1.

The numerator  $|\int G(x)D(x)dx|$  can be expressed as follow:

$$\begin{aligned}
\left| \int GD \right| &= \left| \int \left( -1 + \prod_{k=0}^{\log n} (\gamma f_k + 1) \right) \cdot D \right| \\
&= \left| \int \left( \gamma \sum_{k=0}^{\log n} f_k + \sum_{l=2}^{\log n} \gamma^l \sum_{0 \leq k_1 < \dots < k_l \leq \log n} f_{k_1} \cdots f_{k_l} \right) \cdot D \right| \\
&\geq \gamma \left| \sum_{k=0}^{\log n} \int f_k D \right| - \sum_{l=2}^{\log n} \gamma^l \left| \sum_{0 \leq k_1 < \dots < k_l \leq \log n} \int f_{k_1} \cdots f_{k_l} D \right|.
\end{aligned} \tag{A.3}$$

We consider the integration of a single product  $f_k(x)D(x)$  over a  $k$ -canonical cell  $G_k(i, j)$ . Recall that  $q$  denotes the point in  $P$  that lies in  $G_k(i, j)$ , and  $\chi(q)$  denotes its color. Define vectors  $u = (2^{k-1}, 0)$  and  $v = (0, n/2^{k+1})$ . Then for any point  $x \in G_k(i, j)_{LL}$ , points  $x + u$ ,  $x + v$  and  $x + u + v$  are the analogous points in quadrants  $G_k(i, j)_{LR}$ ,  $G_k(i, j)_{UL}$  and  $G_k(i, j)_{UR}$  of  $x$ , respectively (see Figure 3). Let  $R_x$  denote the orthogonal rectangle defined by  $x$  and  $x + u + v$ , and let function  $R(x)$  be the indicator function of point  $q$  and  $R_x$ , that is,  $R(x) = 1$  if  $q \in R_x$  and  $R(x) = 0$  if otherwise. We can express the integration as

$$\begin{aligned}
\int_{G_k(i, j)} f_k(x)D(x)dx &= \int_{G_k(i, j)_{LL}} \frac{\chi(q)}{n^2} (D(x) - D(x + u) \\
&\quad - D(x + v) + D(x + u + v)) dx \\
&= \int_{G_k(i, j)_{LL}} \frac{\chi(q)}{n^2} \cdot \chi(q)R(x)dx \\
&= \frac{1}{n^2} \int_{G_k(i, j)_{LL}} R(x)dx.
\end{aligned}$$

The second equation is because  $(D(x) - D(x + u) - D(x + v) + D(x + u + v))$  only counts points inside  $R_x$ , which can only be  $q$ , or nothing otherwise. Observe that  $R(x) = 1$  if and only if one of



$x$ 's analogous points lies inside the rectangle defined by  $q$  and its nearest corner (see Figure 3), so we have

$$\begin{aligned} \int_{G_k(i,j)} f_k D &= \frac{1}{n^2} \int_{G_k(i,j)_{LL}} R \\ &= \frac{1}{n^2} V_P(k, i, j). \end{aligned} \quad (\text{A.4})$$

Now we can compute the first term in (A.3):

$$\begin{aligned} \gamma \left| \sum_{k=0}^{\log n} \int f_k D \right| &= \gamma \left| \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} \int_{G_k(i,j)} f_k D \right| \\ &= \gamma \left| \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} \frac{1}{n^2} V_P(k, i, j) \right| \\ &= \frac{\gamma}{n^2} S_P \\ &\geq c\gamma \log n. \end{aligned} \quad (\text{A.5})$$

For the second term in (A.3), recall that the function  $f_{k_1} \cdots f_{k_l}$  is  $(k_1, \log n - k_l)$ -checked. Consider a  $(k_1, \log n - k_l)$ -cell  $G_{k_1, \log n - k_l}$ . Note that  $P$  intersects  $G_{k_1, \log n - k_l}(i, j)$  with at most 1 point. By similar arguments in the proof of equation (A.4), we find the integration  $\left| \int_{G_{k_1, \log n - k_l}(i, j)} f_{k_1} \cdots f_{k_l} D \right|$  is 0 if  $P \cap G_{k_1, \log n - k_l} = \emptyset$  and otherwise equal to the corner volume of  $G_{k_1, k_l}(i, j)$  divide by  $n^2$ . In the latter case, we relax the corner volume to the volume of  $G_{k_1, \log n - k_l}(i, j)$ , that is,  $n/2^{k_l - k_1}$ . Thus we can estimate the integration as follows:

$$\left| \int_{G_{k_1, \log n - k_l}(i, j)} f_{k_1} \cdots f_{k_l} D \right| \leq \frac{1}{n^2} \cdot \frac{n}{2^{k_l - k_1}} = \frac{1}{2^{k_l - k_1} n}.$$

Since there are  $n$  non-empty  $(k_1, \log n - k_l)$ -cells, we have

$$\left| \int f_{k_1} \cdots f_{k_l} D \right| \leq n \cdot \frac{1}{2^{k_l - k_1} n} = \frac{1}{2^{k_l - k_1}}.$$

Now we can estimate the second term in (A.3) :

$$\begin{aligned} \sum_{l=2}^{\log n} \gamma^l \left| \sum_{0 \leq k_1 < \dots < k_l \leq \log n} \int f_{k_1} \cdots f_{k_l} D \right| &\leq \sum_{l=2}^{\log n} \gamma^l \sum_{0 \leq k_1 < \dots < k_l \leq \log n} \frac{1}{2^{k_l - k_1}} \\ &= \sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \sum_{k_l - k_1 = w} \frac{1}{2^w} \binom{w-1}{l-2}. \end{aligned} \quad (\text{A.6})$$

For the last equation we replace  $k_l - k_1$  with a new index  $w$  and use the fact that there are  $\binom{w-1}{l-2}$  ways to choose  $k_2, \dots, k_{l-1}$  in an interval of length  $w$ . Note that for a fixed  $w$ , there are  $\log + 1 - w$

possible values for  $k_1$ , so

$$\begin{aligned}
\sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \sum_{k_l-k_1=w} \frac{1}{2^w} \binom{w-1}{l-2} &= \sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \frac{\log n+1-w}{2^w} \binom{w-1}{l-2} \\
&\leq \sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \frac{\log n}{2^w} \binom{w-1}{l-2} \\
&= \log n \sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \frac{1}{2^w} \binom{w-1}{l-2}. \tag{A.7}
\end{aligned}$$

By inverting the order of the summation,

$$\begin{aligned}
\log n \sum_{l=2}^{\log n} \gamma^l \sum_{w=l-1}^{\log n+1} \sum_{k_l-k_1=w} \frac{1}{2^w} \binom{w-1}{l-2} &= \gamma^2 \log n \sum_{w=1}^{\log n+1} \frac{1}{2^w} \sum_{l=2}^{w+1} \binom{w-1}{l-2} \gamma^{l-2} \\
&= \gamma^2 \log n \sum_{w=1}^{\log n+1} \frac{1}{2^w} (1+\gamma)^{w-1} \\
&= 2\gamma^2 \log n \sum_{w=1}^{\log n+1} \left(\frac{1+\gamma}{2}\right)^{w-1} \\
&\leq \frac{2\gamma^2}{1-\gamma} \log n. \tag{A.8}
\end{aligned}$$

So from (A.5), (A.6), (A.7) and (A.8) we have

$$\left| \int GD \right| \geq c\gamma \log n - \frac{2\gamma^2}{1-\gamma} \log n.$$

Setting  $\gamma$  small enough while combining with (A.1) and (A.2) completes the proof.  $\square$

## A.2 Proof of Lemma 3.2

*Proof.* By Lemma 3.1, we only need to prove that the corner volume sum of any point set  $P \in \mathcal{P}_0$  is large. Fix  $k$  and consider a  $k$ -canonical cell  $G_k(i, j)$ . Let  $q$  denote the point in  $P \cap G_k(i, j)$ . We define the corner  $x$ -distance of  $G_k(i, j)$  to be the difference between the  $x$ -coordinate of  $q$  and that of its nearest corner of  $G_k(i, j)$ . The corner  $y$ -distance is defined in similar manner. See Figure 3. We use  $X(k, i, j)$  and  $Y(k, i, j)$  to denote the corner  $x$ -distance and corner  $y$ -distance, respectively. Note that the corner volume  $V_P(k, i, j)$  is the product of  $X(k, i, j)$  and  $Y(k, i, j)$ .

**Fact A.1.** Fix  $k$  and  $i$ ,  $\{X(k, i, j) \mid j \in [2^k]\} = \{j+1/2, j+1/2 \mid j \in [2^{k-1}]\}$ , where both are taken as multisets.

For a proof, note that the  $k$ -canonical cell  $G_k(i, j)$  is intersecting with  $2^k$  columns:  $G_0(i2^k, 0), \dots, G_0((i+1)2^k - 1, 0)$ . There are  $2^k$  points in  $G_k(i, 0), \dots, G_k(i, 2^k - 1)$ , and they must reside in different columns. Therefore there is exactly one point in the each of the  $2^k$  columns, and their corner  $x$ -distances span from  $1/2$  to  $2^{k-1}/2 - 1/2$ , and each value is hit exactly twice. Similarly, we have

**Fact A.2.** Fix  $k$  and  $j$ ,  $\{X(k, i, j) \mid i \in [n/2^k]\} = \{i + 1/2, i + 1/2 \mid i \in [n/2^{k+1}]\}$ , where both are taken as multisets.

Now consider the product of  $X(k, i, j)$  and  $Y(k, i, j)$  over all  $(i, j)$  for a fixed  $k$ :

$$\begin{aligned}
\prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} V_P(k, i, j) &= \prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} X(k, i, j) Y(k, i, j) \\
&= \prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} X(k, i, j) \cdot \prod_{j=0}^{2^k-1} \prod_{i=0}^{n/2^k-1} Y(k, i, j) \\
&= \prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^{k-1}-1} (j + 1/2)^2 \cdot \prod_{j=0}^{2^k-1} \prod_{i=0}^{n/2^k-1} (i + 1/2)^2.
\end{aligned}$$

The last equation is due to Fact A.1 and Fact A.2. By relaxing  $i + 1/2$  and  $j + 1/2$  to  $(i + 1)/2$  and  $(j + 1)/2$ , we have

$$\begin{aligned}
\prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} V_P(k, i, j) &\geq \prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^{k-1}-1} \left(\frac{j+1}{2}\right)^2 \cdot \prod_{j=0}^{2^k-1} \prod_{i=0}^{n/2^k-1} \left(\frac{i+1}{2}\right)^2 \\
&= \prod_{i=0}^{n/2^k-1} \left(\frac{(2^k-1)!}{2^{2^k-1}}\right)^2 \cdot \prod_{j=0}^{2^k-1} \left(\frac{(n/2^{k+1})!}{2^{n/2^{k+1}}}\right)^2.
\end{aligned}$$

By the inequality  $x! \geq (x/e)^x$ ,

$$\begin{aligned}
\prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} V(k, i, j) &\geq \prod_{i=0}^{n/2^k-1} \left(\left(\frac{2^k-1}{2e}\right)^{2^k-1}\right)^2 \cdot \prod_{j=0}^{2^k-1} \left(\left(\frac{n/2^{k+1}}{2e}\right)^{n/2^{k+1}}\right)^2 \\
&= \prod_{i=0}^{n/2^k-1} \left(\frac{2^k-1}{2e}\right)^{2^k} \cdot \prod_{j=0}^{2^k-1} \left(\frac{n/2^{k+1}}{2e}\right)^{n/2^k} \\
&= \left(\frac{2^k-1}{2e}\right)^{2^k \cdot n/2^k} \cdot \left(\frac{n/2^{k+1}}{2e}\right)^{n/2^k \cdot 2^k} \\
&= \left(\frac{2^k}{4e}\right)^n \cdot \left(\frac{n/2^k}{4e}\right)^n \\
&= \left(\frac{n}{16e}\right)^n.
\end{aligned}$$

Using the inequality of geometric means,

$$\begin{aligned}
\sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} V_P(k, i, j) &\geq n \cdot \left(\prod_{i=0}^{n/2^k-1} \prod_{j=0}^{2^k-1} V_P(k, i, j)\right)^{1/n} \\
&\geq \frac{n^2}{16e}.
\end{aligned}$$

So the corner volume sum  $S_P = \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} V(k, i, j)$  is lower bounded by  $n^2 \log n / 16e$ , and the lemma follows.  $\square$

### A.3 Proof of Lemma 3.4

*Proof.* Note that there are two points of  $P_1 \cup P_2$  in each  $k$ -canonical cell  $G_k(i, j)$ , and we let  $q_1, q_2$  denote the two points from  $P_1$  and  $P_2$ , respectively. We will set  $f_k(x) = C/n^2$  for quadrants  $G_k(i, j)_{UR}$  and  $G_k(i, j)_{LL}$  and  $f_k(x) = -C/n^2$  for the other two quadrants, where  $C$  is determined as follows:

$$C = \begin{cases} \chi(q_1) & \text{if } V_{P_1}(k, i, j) \geq V_{P_2}(k, i, j); \\ \chi(q_2) & \text{if } V_{P_1}(k, i, j) < V_{P_2}(k, i, j). \end{cases}$$

Let  $D(x)$  be the combinatorial discrepancy at  $x$  over  $P_1 \cup P_2$ . By similar argument in the proof of Lemma 3.1, we get

$$\int_{G_k(i, j)} f_k D = \begin{cases} \frac{1}{n^2} (V_{P_1}(k, i, j) + V_{P_2}(k, i, j)) & \text{if } \chi(q_1) = \chi(q_2); \\ \frac{1}{n^2} |V_{P_1}(k, i, j) - V_{P_2}(k, i, j)| & \text{if } \chi(q_1) \neq \chi(q_2). \end{cases}$$

In either case,

$$\int_{G_k(i, j)} f_k D \geq \frac{1}{n^2} |V_{P_1}(k, i, j) - V_{P_2}(k, i, j)|.$$

And the rest of the proof follows the same argument in the proof of Lemma 3.1. □