On the Cell Probe Complexity of Dynamic Membership

or

Can We Batch Up Updates in External Memory?

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The power of buffering

- For numerous dynamic data structure problems in external memory, updates can be buffered.
  - Buffer tree [Arge 1995]
  - Logarithmic method [Bentley 1980] $+$ B-tree

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problem	update	query	cache-oblivious
stack	$O(1/b)$	/	trivial
queue	$O(1/b)$	/	trivial
priority-queue	$O(\frac{1}{b}\log_b n)$	/	[Arge et. al. STOC 02]
predecessor	$O(\frac{1}{b}\log n)$	$O(\log n)$	trivial
range-sum range-reporting $\ldots$	$O(\frac{b^{\epsilon}}{b}\log n)$	$O(\log_b n)$	[Brodal et. al. this SODA]

b: size of a block/cell (in words)

How about Dictionary and Membership?

- Dictionary and membership (selected)
  - Knuth, 1973: External hashing
    Expected average cost of an operation is $1 + 1/2^{\Omega(b)}$, provided the load factor α is less than a constant smaller than 1. (truly random hash function)
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  - Data structures like Arge's Buffer tree:
    Update $= O(\frac{b^\epsilon}{b} \log n)$, Query $= O(\log_b n)$.
- **Question**: can we improve the amortized update cost to $o(1)$ in external memory, without sacrificing the query speed by much?

The conjecture

A long-time folklore conjecture in external memory community: (explicitly stated by Jensen and Pagh, 2007)

t_u must be $\Omega(1)$ if t_q is required to be $O(1)$

t_u: expected amortized update cost

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Our small step: ≤ 1.1

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t_q: expected average query cost

Problems

Membership: Maintain a set $S \subseteq U$ with $|S| \leq n$.
Given an $x \in U$, is $x \in S$? Yes or No.

Dictionary: If $x \in S$, return associated info, otherwise say No. Often assumes "indivisibility".

Objective: Tradeoff between update cost t_u and query cost t_q

Two of the most fundamental data structure problems in computer science!

The computational model

- The cell probe model [Yao 1981] with a content preserving cache
  - A data structure is a collection of b-bit cells
  - Cost of an operation: $\#$ of cells read/changed
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  Our results hold for arbitrary b, though they are more meaningful for large b's
- The cache may not affect t_q by much, but does affect t_u in almost all common data structures (typically $o(1)$).

Let's go!

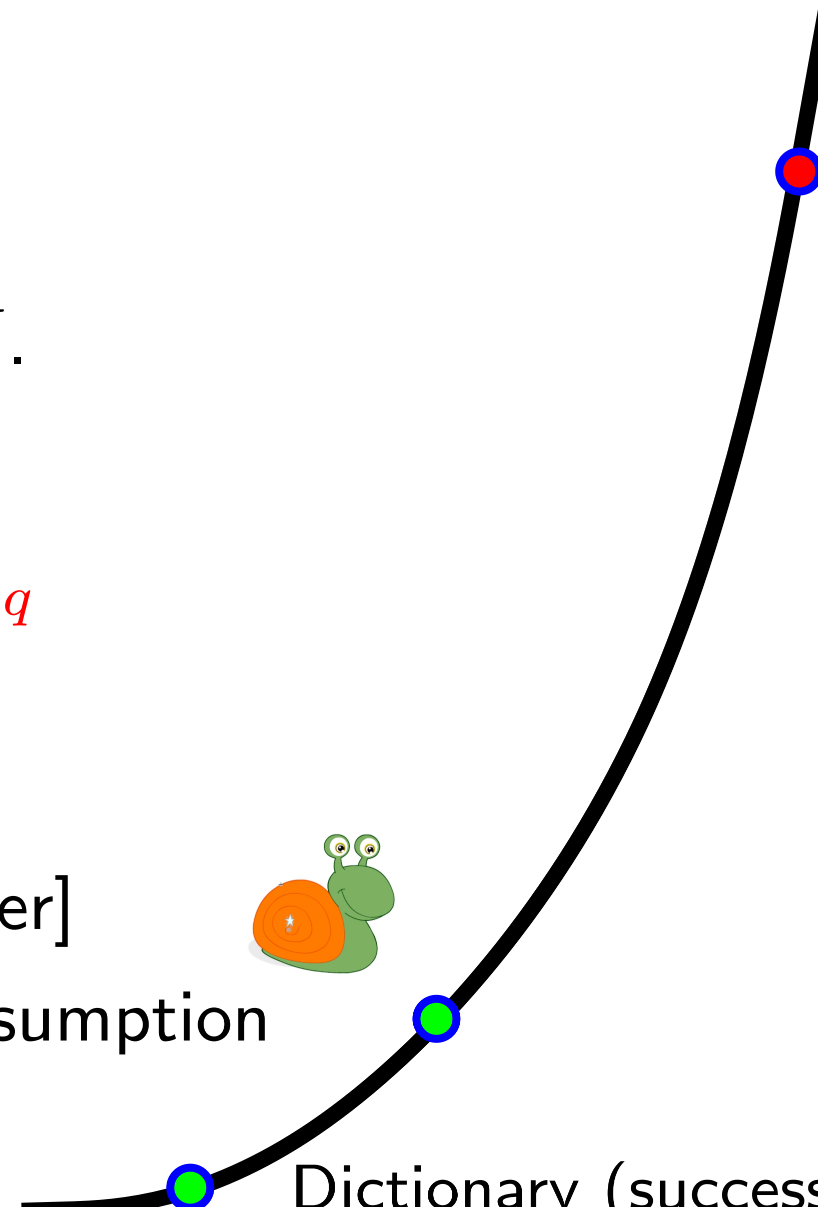
Membership

Problem: Maintain a set $S \subseteq U$.
Given $x \in U$, is $x \in S$?

Goal: tradeoff between t_u and t_q

Membership $t_q = 1 + \delta$
$(0 \leq \delta < 1/2)$ [this paper]

Without indivisibility assumption



Dictionary (successful)
[SPAA 09] Wei, Yi and Zhang

Outline

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- Future work

Preliminaries

- $U = \{0, 1, \ldots, u-1\}$: universe. $|U| = u$.

- m: size of cache. In bits.

  b: size of one cell. In bits.

  n: total number of inserted elements.

- S: set of elements we are maintaining. $|S| \leq n$

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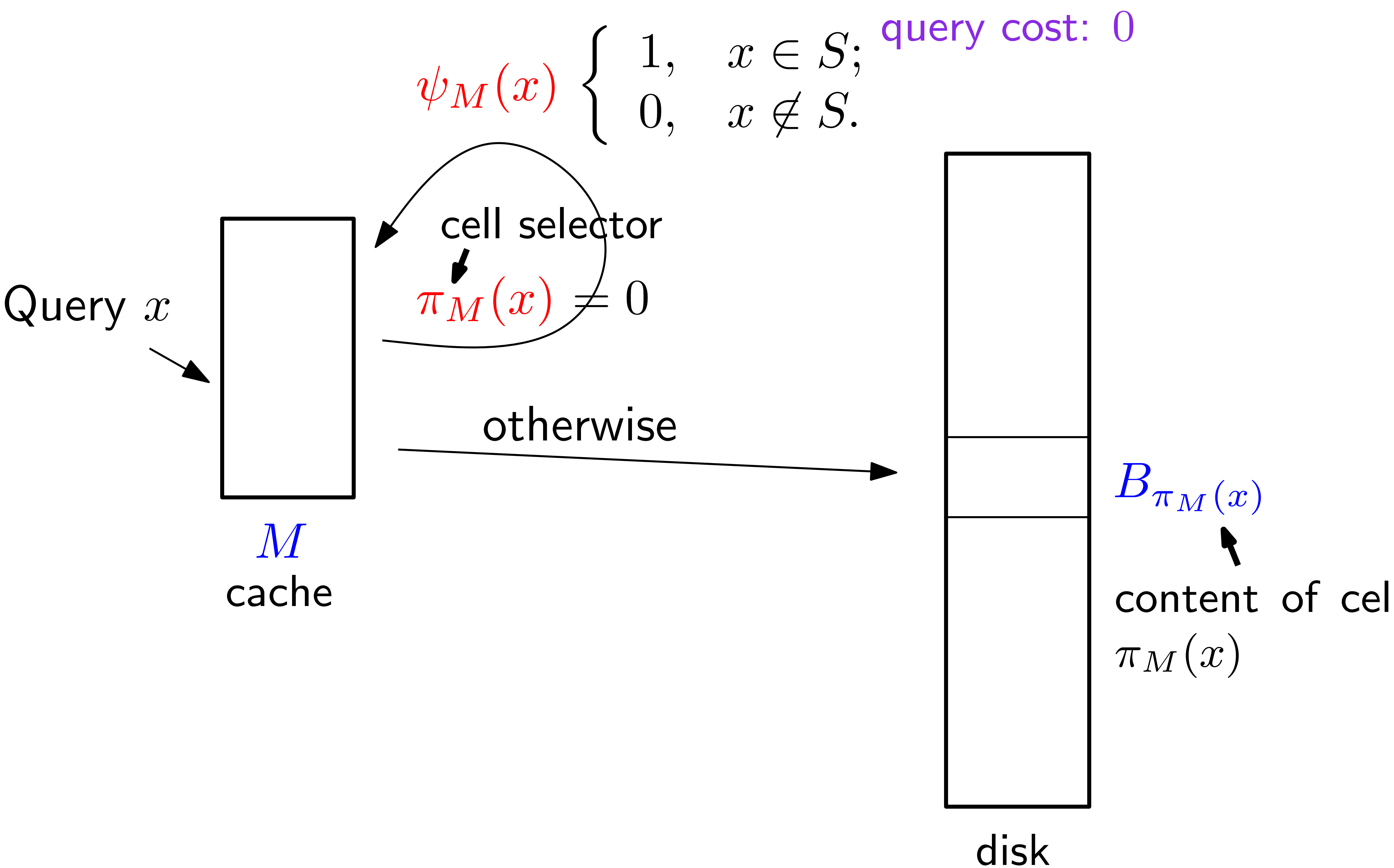
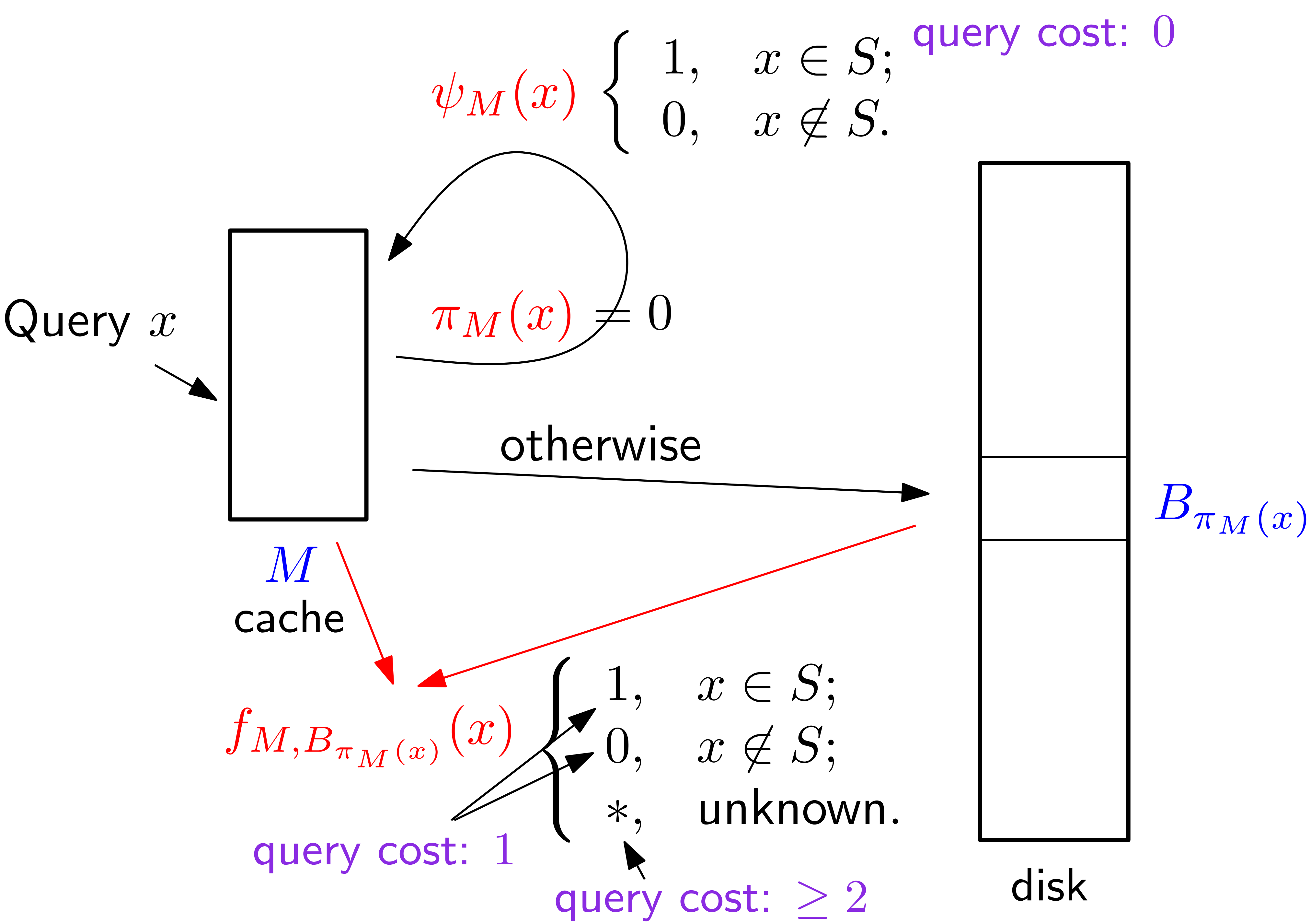
- m: size of cache. In bits.
  b: size of one cell. In bits.
  n: total number of inserted elements.
- S: set of elements we are maintaining. $|S| \leq n$
- A very mild assumption
  - $u \geq \Omega(n) \geq \Omega(mb)$

The model



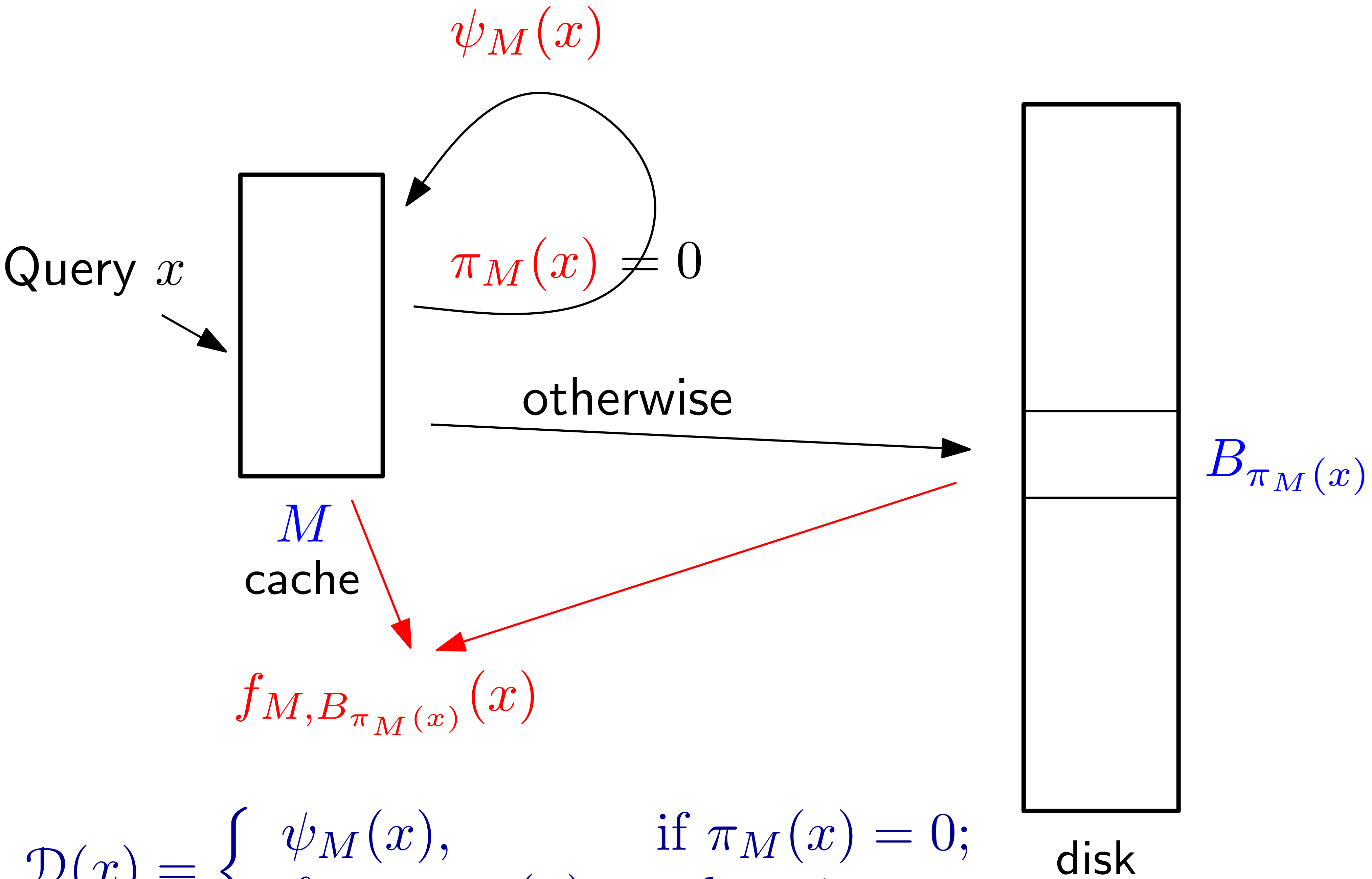
$\psi_M(x) \begin{cases} 1, & x \in S; \\ 0, & x \notin S. \end{cases}$
query cost: 0
cell selector
$\pi_M(x) = 0$
Query x
otherwise
M
cache
$B_{\pi_M(x)}$
content of cell $\pi_M(x)$
disk

The model



query cost: 0
$\psi_M(x)$ $\begin{cases} 1, & x \in S; \\ 0, & x \notin S. \end{cases}$
Query x
$\pi_M(x) \neq 0$
otherwise
$B_{\pi_M(x)}$
M
cache
$f_{M,B_{\pi_M(x)}}(x)$ $\begin{cases} 1, & x \in S; \\ 0, & x \notin S; \\ *, & \text{unknown.} \end{cases}$
query cost: 1
query cost: ≥ 2
disk

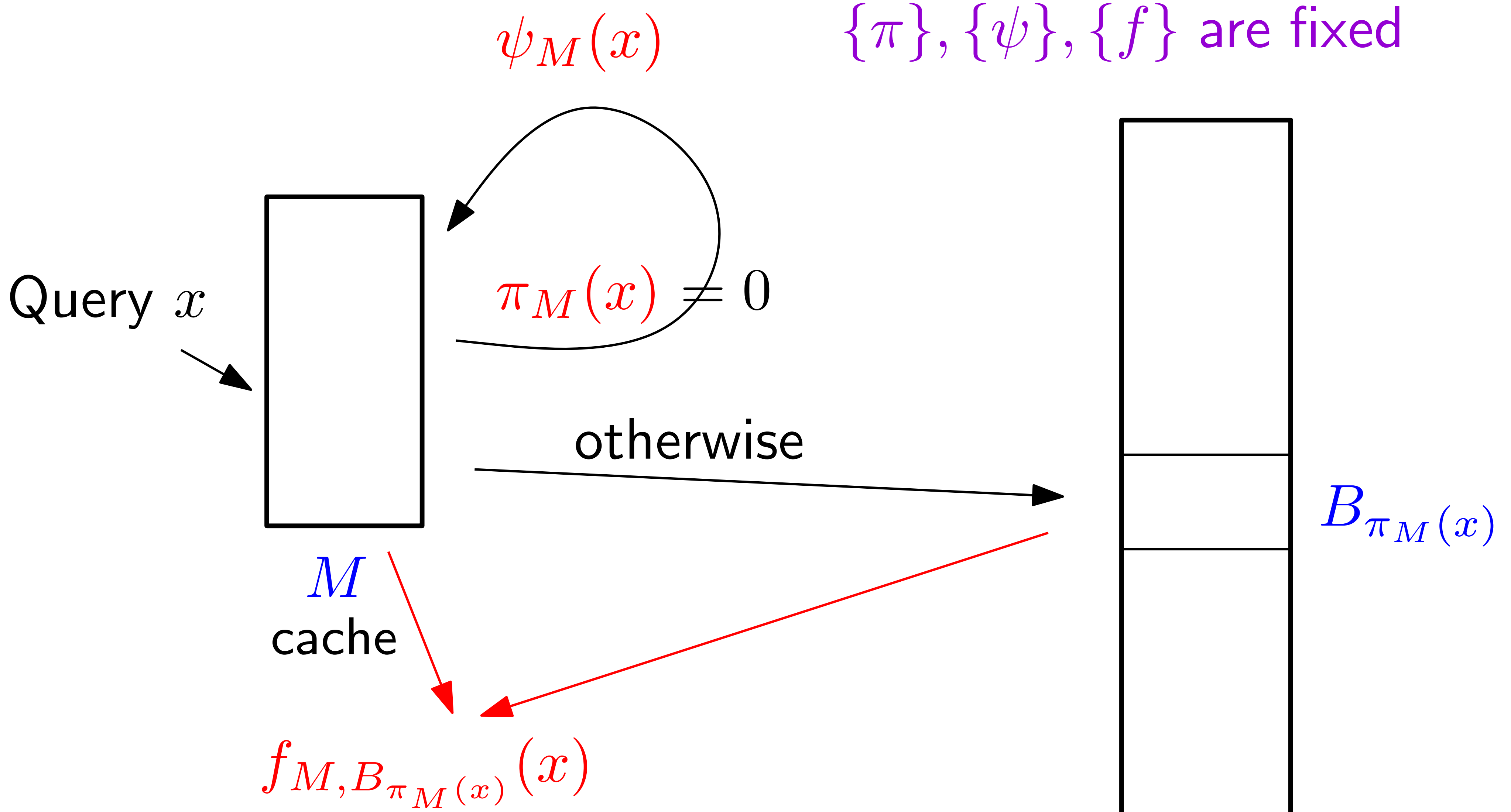
The model



$$\mathcal{D}(x) = \begin{cases} \psi_M(x), & \text{if } \pi_M(x) = 0; \\ f_{M,B_{\pi_M(x)}}(x), & \text{otherwise.} \end{cases}$$

The model

Families of functions $\{\pi\}, \{\psi\}, \{f\}$ are fixed



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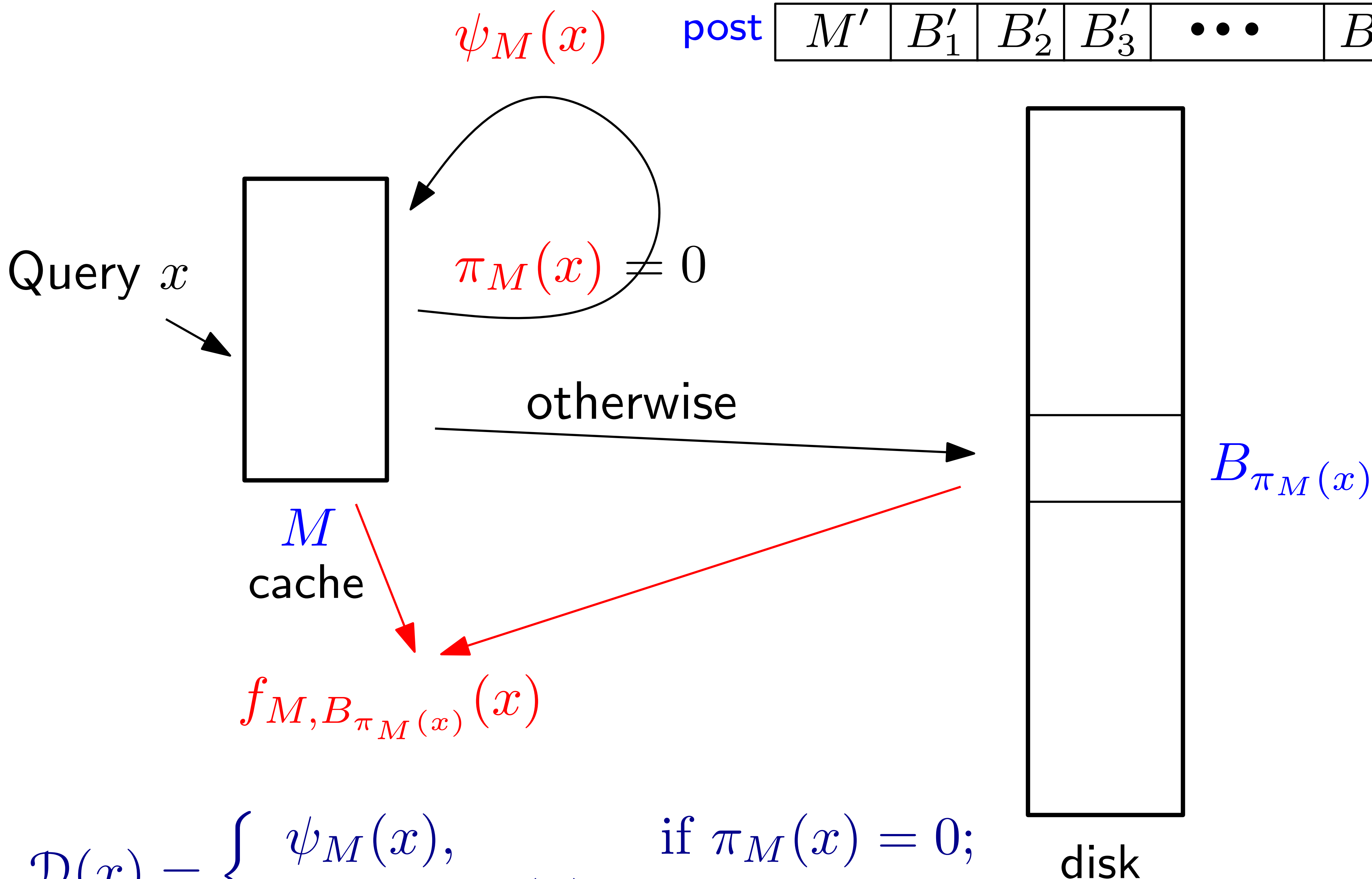
The model

During an update

pre	M	B_1	B_2	B_3	$\bullet\bullet\bullet$	B_d
post	M'	B'_1	B'_2	B'_3	$\bullet\bullet\bullet$	B'_d



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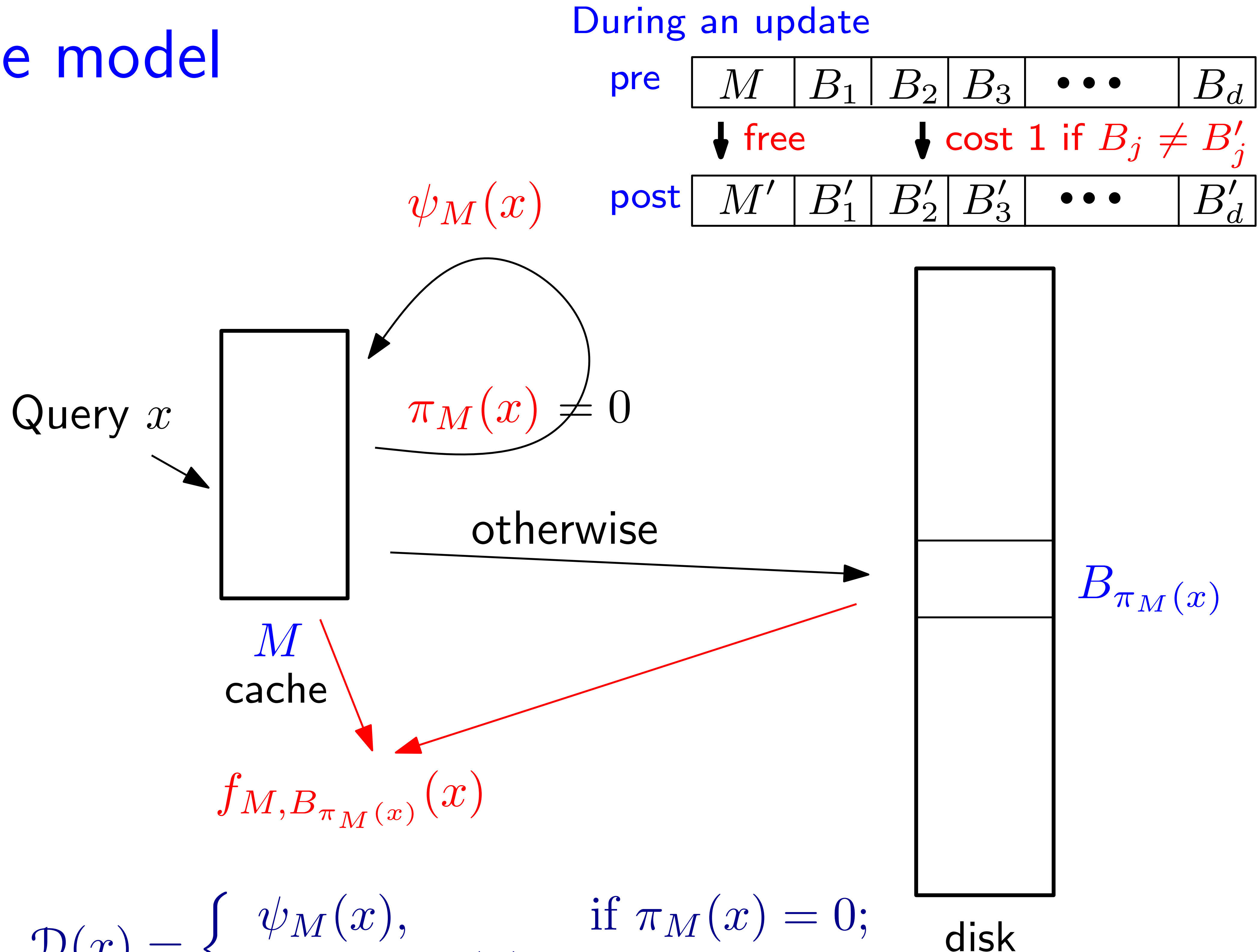

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Framework of the proof

- During the insertion sequence,
  1. neglect first σn elements,
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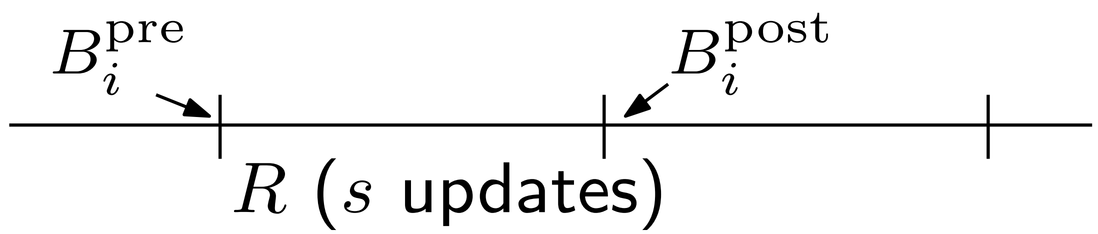

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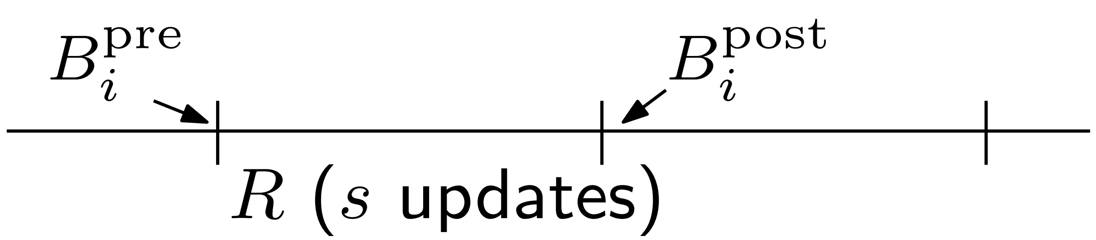

- Cells i having $B_i^{\text{pre}} \neq B_i^{\text{post}}$ must be modified in round R.

We try to show that during each round:

at least $\Omega(s)$ cells i have $B_i^{\text{pre}} \neq B_i^{\text{post}}$.

$\longrightarrow$ amortized update cost is $\Omega(1)$

High level ideas of the proof (focus on 1 round)

Consider queries at the final snapshot of a round.

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3 (because of 2). Cell selector $\pi(\cdot)$ used has to be balanced.

Intuition: otherwise the data structure will not be correct,
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Let $\alpha_i = |\{x \mid \pi(x) = i\}|/u$. $\pi(\cdot)$ is balanced if
there are not too many $\alpha_i \geq \Omega\left(\frac{b}{n}\right)$

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$1 + 3 \Rightarrow 4$. In a round, inserted elements' query paths go to many different cells after probing the cache.

High level ideas of the proof (cont.)

5. $\Omega(s)$ cells have to change.

   Intuition: new elements are chosen randomly from U. For cell i, no matter what B_i^{pre} is, if $\{f_{M,B_i^{\mathrm{post}}}(x) \mid \pi_M(x) = i\}$ contains few "$*$", then $B_i^{\mathrm{pre}} \neq B_i^{\mathrm{post}}$ with high probability.

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Finally,

- (2) – (5) hold with high probability $\left(1 - e^{-\Omega(n)}\right)$, therefore hold for all 2^m states of M w.h.p.
- Total cost per round is $\Omega(s)$
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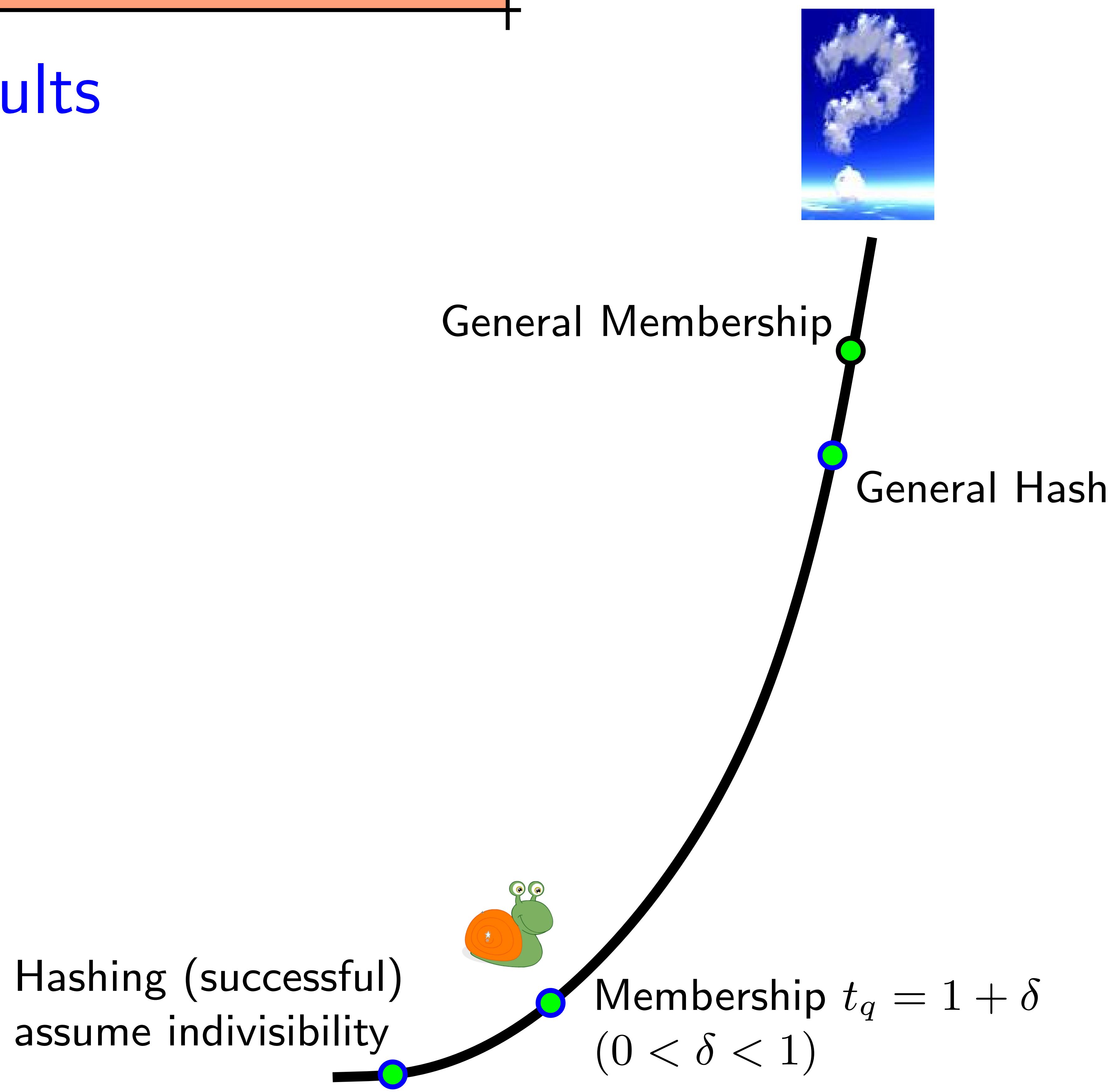
Finished

Latest results



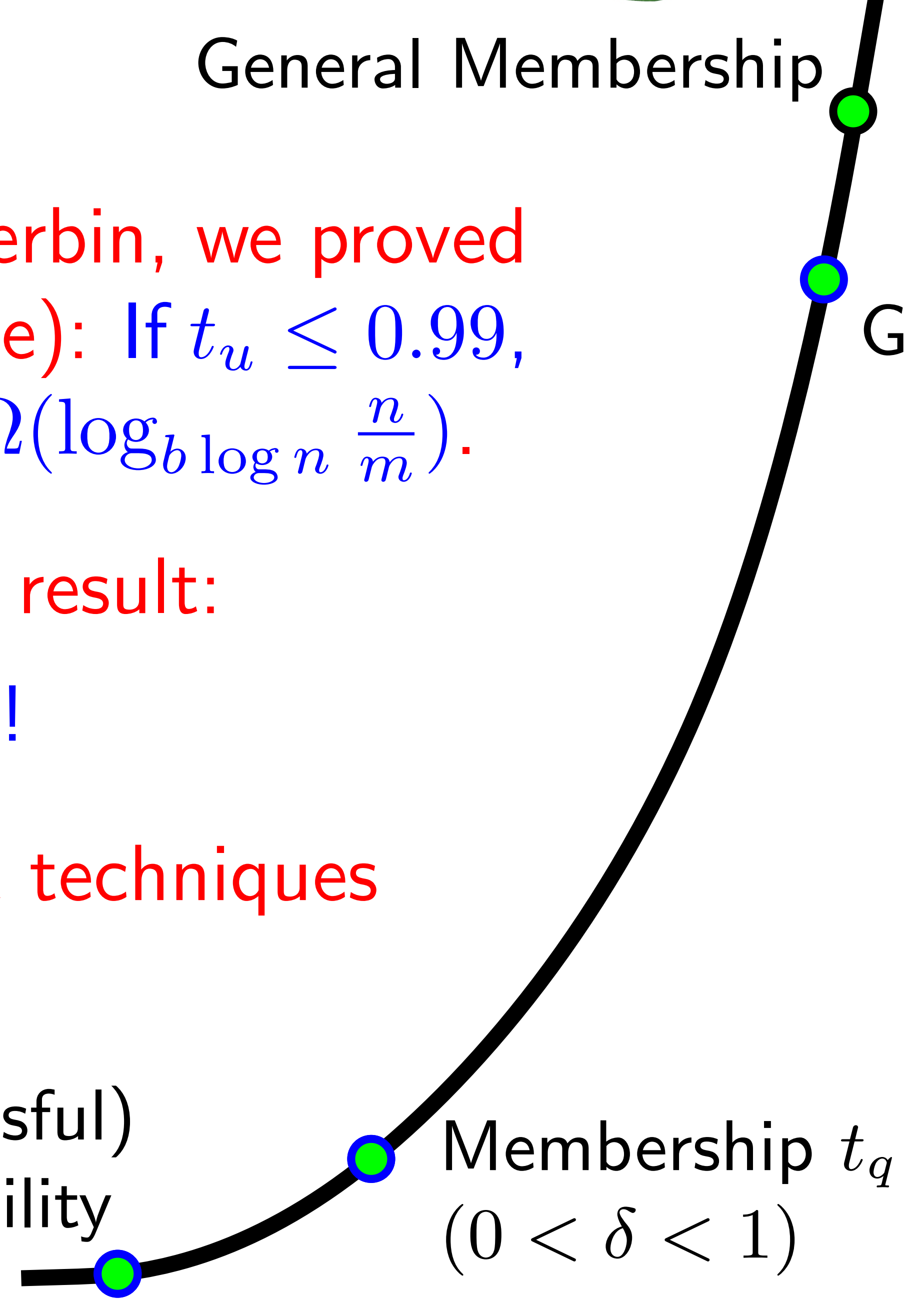
Latest results

Very recently with Elad Verbin, we proved this conjuecture (even more): If $t_u \leq 0.99$, then t_q is required to be $\Omega(\log_{b \log n} \frac{n}{m})$.

- A strong dichotomy result:

  Hash or Buffer-tree !

- Completely different techniques



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- Can we simplify the complicated combinatorial proof?
  Use, e.g., encoding arguments like Pătrașcu-Viola.

The End

THANK YOU

Q and A